Pf 2 of Lemma 2.51

$I_f := \{ b \in \Gamma(V) \mid f \cdot b \in \Gamma(V) \}$ is a nonzero ideal of $\Gamma(V)$ and $V(I_f) \subsetneq V$ is the set of points $P \in V$ at which f is not defined. $\square$

Any $f \in K(V)$ gives rise to a map $f : U_f \longrightarrow K$.

Lemma 2.52 If $U, U' \neq \emptyset$ are open subsets of an irreducible alg. subset $V \subseteq K^n$, then $U \cap U' \neq \emptyset$.

Pf $V \setminus U$ and $V \setminus U' \subsetneq V$ are closed subsets of V. If $U \cap U' = \emptyset$, then $V = (V \setminus U) \cup (V \setminus U')$, so V is reducible. $\square$

Lemma 2.53 If $f \in K(V)$ is zero on a nonempty open subset $U \subseteq U_f$, then $f = 0$.

Pf Write $f = \frac{a}{b}$. For any $P \in U$, we have $b(P) = 0$ or $a(P) = 0$.

$$\Rightarrow U \cap (V \setminus V(a)) \cap (V \setminus V(b)) = \emptyset \Rightarrow \text{↯ by Lemma 2.52}$$

nonempty open subsets of V if $a \neq 0$

$\square$

Cor If $f, g \in K(V)$ agree on a nonempty open subset $U \subseteq U_f \cap U_g$, then $f = g$.

Rmk This is similar to facts from complex analysis: If two meromorphic functions $f, g : \mathbb{C} \to \mathbb{C}$ agree on a nonempty open subset of $\mathbb{C}$, then $f = g$.

Cor The elements of $K(V)$ correspond bijectively to pairs (U, f) with $\emptyset \neq U \subseteq V$ open and $f : U \to K$ ~~any~~ map, which is locally given by a quotient of regular functions

$$\left(\forall P \in U \ \exists P \in U' \subseteq U \text{ open}, \quad a, b \in \Gamma(V) : \quad \forall Q \in U' : \ b(Q) \neq 0, \ f(P) = \frac{a(P)}{b(P)} \right),$$

where we identify $(U, f), (U', f')$ if

$$f|_{U \cap U'} = f'|_{U \cap U'}.$$

Reminder

The field of fractions of an integral domain is the set of pairs (a, b) with $a, b \in R$, $b \neq 0$, where we identify (a, b) and (a', b') if $ab' = a'b$.

((a, b) corresponds to $\frac{a}{b}$.)

Rmk If $\varphi: V \to W$ is a dominant morphism, $(\overline{\varphi(V)} = W)$
then we obtain an injective ring homomorphism

$$\varphi^* : \Gamma(W) \hookrightarrow \Gamma(V)$$
$$f \mapsto f \circ \varphi$$

which induces a field homomorphism

$$\varphi^* : K(W) \longrightarrow K(V)$$
$$\frac{a}{b} \longmapsto \frac{\varphi^*(a)}{\varphi^*(b)}$$
$$f \longmapsto f \circ \varphi$$

Rmk Dominance is important:
Otherwise, f might not be defined at any point in $\varphi(V)$, so $f \circ \varphi$ wouldn't be defined at any point in V!

Rmk We have $\underbrace{U_{\varphi^*(f)}}_{\text{(open subset of } V)} \supseteq \varphi^{-1}(\underbrace{U_f}_{\neq \emptyset, \text{ open subset of } W}) \neq \emptyset$ $\boxed{\varphi \text{ dominant}}$

<u>Def</u> The <u>local ring</u> of V at $P \in V$ is

$$\mathcal{O}_{V,P} := \{ f \in k(V) \text{ defined at } P \}.$$

(Fulton denotes it by $\mathcal{O}_P(V)$.)

<u>Thm 2.54</u>

a) $\mathcal{O}_{V,P}$ is a <u>local ring</u> (ring with exactly one maximal ideal) with maximal ideal $m_{V,P} = \{ f \in \mathcal{O}_{V,P} \mid f(P) = 0 \}$.

b) We have
$$\mathcal{O}_{V,P} / m_{V,P} \cong k .$$
$$f \mapsto f(P)$$

<u>Pf</u> <u>ring</u>: if $b_1(P), b_2(P) \neq 0$, then $\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1 b_2 + a_2 b_1}{b_1 b_2}$

with $(b_1 b_2)(P) \neq 0$, ...

<u>ideal</u>: clear

<u>map in b surj</u>: constant fct.

inj: clear

$\Rightarrow$ max. ideal ✓

only max. ideal: Let $I \underset{\neq}{\subseteq} \mathcal{O}_{V,P}$ be another maximal ideal. $\Rightarrow I \not\subseteq m_{V,P}$. Let $f \in I \setminus m_{V,P}$. $\Rightarrow f(P) \neq 0$.

$\Rightarrow \frac{1}{f}$ defined at $P \Rightarrow \frac{1}{f} \in \mathcal{O}_{V,P}$

$\overset{\uparrow}{\Rightarrow}$ $1 = f \cdot \frac{1}{f} \in I \Rightarrow I = \mathcal{O}_{V,P}$ ↯

I ideal

□

<u>Exe</u> $V = K$, $P = c \in K = V$

$\Rightarrow \Gamma(V) = K[X]$, $K(V) = K(X)$

$$\mathcal{O}_{V,P} = \left\{ \frac{a}{b} \;\middle|\; a, b \in K[X],\; b(c) \neq 0 \right\}$$

$\Updownarrow$

b not divisible by $X-c$

$$m_{V,P} = \left\{ \frac{a}{b} \;\middle|\; a(c) = 0,\; b(c) \neq 0 \right\}$$

$=$ ideal of $\mathcal{O}_{V,P}$ generated by $X-c$.

<u>Rmk</u> If $\varphi : V \to W$ is any morphism, and $P \in V$, then we obtain a ring homomorphism

$$\begin{aligned} \varphi^* : \mathcal{O}_{W,\varphi(P)} &\longrightarrow \mathcal{O}_{V,P} \\ \frac{a}{b} &\longmapsto \frac{\varphi^*(a)}{\varphi^*(b)} \\ f &\longmapsto f \circ \varphi \end{aligned}$$

with $\varphi^*(m_{W,\varphi(P)}) \subseteq m_{V,P}$.

Thm 2.55 $\Gamma(V) = \bigcap_{P \in V} \mathcal{O}_{V,P}$

($f \in \Gamma(V) \Leftrightarrow$ f defined at every $P \in V$)

Pf "$\Rightarrow$" clear

"$\Leftarrow$" Let $I_f = \{ b \in \Gamma(V) \mid f \cdot b \in \Gamma(V) \}$ as before.

$$\underset{\substack{\uparrow \\ f \text{ defined everywhere}}}{\Rightarrow} V(I_f) = \emptyset \underset{\substack{\uparrow \\ \text{weak} \\ \text{Nullstellensatz}}}{\Longrightarrow} 1 \in I_f \Rightarrow f \in \Gamma(V).$$

$\square$

Lemma 2.56 The ring $\mathcal{O}_{V,P}$ is noetherian.

Pf Let I be an ideal of $\mathcal{O}_{V,P}$.

$\Rightarrow I \cap \Gamma(V)$ is an ideal of $\Gamma(V)$.

$\Rightarrow I \cap \Gamma(V) = (g_1, \ldots, g_m)_{\Gamma(V)}$ for some $g_1, \ldots, g_m \in \Gamma(V)$.

↑ The quotient $\Gamma(V)$ of $K[x_1, \ldots, x_n]$ is noetherian.

Let $f = \frac{a}{b} \in I$, $b(P) \neq 0$.

$\Rightarrow a = f \cdot b \in I \cap \Gamma(V) = (g_1, \ldots, g_m)$

$\Rightarrow \frac{a}{b} \in (g_1, \ldots, g_m)_{\mathcal{O}_{V,P}}$. $\square$

$\frac{1}{b} \in \mathcal{O}_{V,P}$

Def For any open subset $U \subseteq V$, $\mathcal{O}_V(U)$ is the ring of rational functions $f \in K(V)$ defined at every point $P \in U$.

Rmk If $\varphi: V \to W$ is any morphism between irreducible alg. sets and $U \subseteq W$ open, we obtain a ring homomorphism $\varphi^*: \mathcal{O}_W(U) \longrightarrow \mathcal{O}_V(\varphi^{-1}(U))$.

Def Let $V \subseteq K^n$ and $W \subseteq K^m$ be algebraic subsets. A rational map $\varphi: V \dashrightarrow W$ is a pair (U, φ), where $\emptyset \neq U \subseteq V$ is open, and $\varphi: U \to W$ is a map given by rational functions $f_1, \dots, f_m \in \mathcal{O}_V(U)$:

$$\varphi(P) = (f_1(P), \dots, f_m(P)) \ \forall P \in U,$$

where we identify $(U, \varphi), (U', \varphi')$ if

$$\varphi|_{U \cap U'} = \varphi'|_{U \cap U'}.$$

Rmk A rat. map $V \to K^m$ "is" a tuple $(f_1, \dots, f_m)$ of rational functions $f_1, \dots, f_m \in K(V)$.

Ex Any morphism $\varphi: V \to W$ is a rational map.

Def If $\varphi: A \dashrightarrow B$ and $\psi: B \dashrightarrow C$ are
(def. on U) (def. on U')

rational maps and φ is dominant we get a composition

$$\psi \circ \varphi : A \dashrightarrow C$$

(def. on $\underbrace{\varphi^{-1}(U')}_{\text{nonempty open subset of } A \text{ if } \varphi \text{ is dominant}}$)

Rmk If $\varphi: V \dashrightarrow W$ is dominant ($\overline{\varphi(U)} = W$), we get a field homomorphism

$$\varphi^*: K(W) \longrightarrow K(V).$$
$$f \longmapsto f \circ \varphi$$

Rmk We get a bijection

$$\left\{ \begin{matrix} \text{dominant rational} \\ \varphi: V \dashrightarrow W \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{field hom.} \\ \varphi^*: K(W) \to K(V) \end{matrix} \right\}$$

Pf Same as for morphisms:
$f_i = \varphi^*(X_i)$, where $X_i \in \Gamma(W)$ is the

rational map sending any point P to its i-th coordinate. $\square$

<u>Def</u> V, W are <u>birational</u> if there are dominant rational maps $\varphi: V \dashrightarrow W$ and $\psi: W \dashrightarrow V$ such that $\psi \circ \varphi = id_V$ and $\varphi \circ \psi = id_W$.

<u>Ex</u> $V(X^3 - Y^2) \subset K^2$ is not isomorphic to K (see problem 1d on problem set 3).
But they are birational:

$$\varphi: K \longrightarrow V(X^3 - Y^2)$$
$$t \longmapsto (t^2, t^3)$$

$$\psi: V(X^3 - Y^2) \dashrightarrow K$$
$$(x, y) \longmapsto \frac{y}{x}$$
(defined on $V(X^3 - Y^2) \setminus \{0, 0\}$)

<u>Rmk</u> V, W are birational if and only if the fields $K(V), K(W)$ are isomorphic.