Pf "$\Rightarrow$" Apply Cor 2.48 to $S(f,g) \in I$.

"$\Leftarrow$" Let $0 \neq f \in I$. Write

$$f = \lambda_1 g_1 H_1 + \dots + \lambda_r g_r H_r \qquad (I)$$

with $0 \neq g_i \in G$ and monomials $H_i \in S$ with minimal $M := \max_{1 \le i \le r} (\mathrm{lm}(g_i H_i))$. $\lambda_i \in K^\times$

Clearly $\mathrm{lm}(f) \le M$.

If $\mathrm{lm}(f) = M$, then $\mathrm{lm}(f) = \mathrm{lm}(g_i H_i)$
$= \mathrm{lm}(g_i) \cdot H_i$,
so $\mathrm{lm}(f)$ is divisible by the leading mon. of an element of G.

Assume $\mathrm{lm}(f) < M$.

Since the monomial M has to cancel in the RHS of (I).

w.l.o.g. $\mathrm{lm}(g_i H_i) = M$ for $i = 1, \dots, t$
$\mathrm{lm}(g_i H_i) < M$ for $i = t+1, \dots, r$

$$\Rightarrow \sum_{i=1}^{t} \lambda_i \, \mathrm{lc}(g_i) = 0. \quad (\text{in part., } t \ge 2)$$

By assumption, we can write

$$\frac{M}{\mathrm{lcm}(\mathrm{lm}(g_i), \mathrm{lm}(g_1))} \cdot S(g_i, g_1) = \frac{M}{\mathrm{lt}(g_i)} \cdot g_i - \frac{M}{\mathrm{lt}(g_1)} \cdot g_1$$

$$= \sum_j p_j^{(i)} \cdot q_j^{(i)}$$

with $0 \neq p_j^{(i)} \in G$ and $q_j^{(i)} \in K[x_1, \dots, x_n]$,

and $\mathrm{lm}(p_j^{(i)} \cdot q_j^{(i)}) \leq \mathrm{lm}\left(\frac{M}{\mathrm{lt}(g_i)} g_i - \frac{M}{\mathrm{lt}(g_1)} \cdot g_1\right)$

$< M$.

$$\Rightarrow g_i H_i = \frac{\mathrm{lt}(g_i) H_i}{M} \cdot \sum p_j^{(i)} q_j^{(i)} + \frac{\mathrm{lt}(g_i) H_i H_1}{\mathrm{lt}(g_1) H_1} \cdot g_1 \qquad \text{for } i = 1, \dots, t$$

$$= \mathrm{lc}(g_i H_i) \cdot \sum p_j^{(i)} q_j^{(i)} + \frac{\mathrm{lc}(g_i) H_1}{\mathrm{lc}(g_1)} \cdot g_1$$

$$\Rightarrow \lambda_1 g_1 H_1 + \dots + \lambda_t g_t H_t$$

$$= \sum_{i=1}^{t} \underbrace{\lambda_i \mathrm{lc}(g_i H_i)}_{\in K} \cdot \underbrace{\sum \overbrace{p_j^{(i)}}^{\in G} q_j^{(i)}}_{\mathrm{lm}(\cdot) < M} + \underbrace{\sum_{i=1}^{t} \frac{\lambda_i \mathrm{lc}(g_i)}{\mathrm{lc}(g_1)}}_{0} \cdot g_1 H_1$$

$\Rightarrow$ We can rewrite f as a sum as in (I)

with smaller $M = \max_{1 \leq i \leq r} (\mathrm{lm}(g_i H_i))$. $\lightning$

$\square$

<u>Similar to:</u>

$\{(a_1, \dots, a_n) \mid a_1 + \dots + a_n = 0\}$ is spanned by

$e_i - e_j$ for $1 \leq i, j \leq n$.

Buchberger's Algorithm

We can compute a finite Gröbner basis of $I = (f_1, \dots, f_m)$ as follows:

Construct sets

$$F = G_0, \; G_1, \; G_2, \; \dots$$

of polynomials generating I such that

$$(\mathrm{lm}(G_0)) \subsetneq (\mathrm{lm}(G_1)) \subsetneq (\mathrm{lm}(G_2)) \subsetneq \cdots$$

If G_k fails Buchbergers Criterion, there is a <u>reduction</u> $r \neq 0$ of some $S(g_1, g_2)$ with $g_1, g_2 \in G_k$ (w.r.t. G_k).

$\Rightarrow$ $\mathrm{lm}(r)$ is not divisible by any element of $\mathrm{lm}(G_k)$.

Take $G_{k+1} = G_k \cup \{r\}$.

$\Rightarrow$ $(\mathrm{lm}(G_{k+1})) \supsetneq (\mathrm{lm}(G_k))$.

By Hilbert's Basis Theorem, this process terminates after a finite number of steps.

<u>Rmk</u> You can also after each step replace any elements of G_k by its reduction w.r.t. $G_k \setminus \{g\}$, one polynomial g at a time.

Ese $I = (XY^2, X^2Y+1)$, lex. order

$f_1 = XY^2$ $\quad G_0 = \{f_1, f_2\}$

$f_2 = X^2Y+1$

$r = S(f_1, f_2) = X \cdot f_1 - Y \cdot f_2 = -Y$

is reduced w.r.t. $\{f_1, f_2\}$.

$G_1 = \{\cancel{f_1}, f_2, r\}$

$f_2 + X^2 \cdot r = 1$

$G_1' = \{1\}$.

is a Gröbner basis

Ese $I = (X^3-2XY, X^2Y-2Y^2+X)$, deg. lex. order

$f_1 = X^3 - 2XY$

$f_2 = X^2Y - 2Y^2 + X$

$r = S(f_1, f_2) = Y \cdot f_1 - X \cdot f_2 = -2XY^2 + 2XY^2 - X^2$

$= -X^2$

is reduced w.r.t. $\{f_1, f_2\}$

$G_1 = \{f_1, f_2, r\}$

$f_1' = f_1 + X \cdot r = -2XY$

$G_1' = \{f_1', f_2, r\}$

$f_2' = f_2 + Y \cdot r = -2Y^2 + X$

$G_1'' = \{f_1', f_2', r\}$

$S(f_1', f_2') = Y \cdot f_1' - X \cdot f_2' = -X^2$

reduces to 0 w.r.t. $\{f_1', f_2', r\}$

$S(f_1', r) = X \cdot f_1' - 2Y \cdot r = 0$

$S(f_2', r) = X^2 \cdot f_2' - 2Y^2 \cdot r = X^3$

reduces to 0 w.r.t. $\{f_1', f_2', r\}$

$\Rightarrow$ $\{f_1', f_2', r\}$ is a Gröbner basis.

Often, deg. rev. lex. order is faster than lex. order.

Another aside

Thm Let $f \in K[X_1, \dots, X_n]$ and assume that

$$V\left(f, \frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right) = \emptyset.$$

(There is no $P \in K^n$ with $f(P) = \frac{\partial f}{\partial X_1}(P) = \dots = \frac{\partial f}{\partial X_n}(P) = 0$.)

Then, (f) is a radical ideal. (f is squarefree.)

Eg $X^2 + Y^2 - 1$ is squarefree if $\mathrm{char}(K) \neq 2$:

$$V(X^2+Y^2-1, 2X, 2Y) = \emptyset.$$

Warning The theorem is not an equivalence!

Pf of Thm Assume $f = g^2 h$, where g is a nonconstant polynomial and h is any polynomial.

Then $$\frac{\partial f}{\partial X_i} = g^2 \frac{\partial h}{\partial X_i} + 2g \frac{\partial g}{\partial X_i} h.$$

Let $P \in V(g)$. Then, $\frac{\partial f}{\partial X_i}(P) = 0$. $\square$

2.13. Rational functions

Let $V \subseteq K^n$ be an irreducible variety.
Recall that this means that $\Gamma(V)$ is an integral domain.

<u>Def</u> The <u>field of rational functions</u> on V is the field of fractions $K(V)$ of $\Gamma(V)$.

<u>Ex</u> $V = K^n \rightsquigarrow K(V) = K(X_1, \dots, X_n)$.

<u>Ex A</u> $V = V(XY - Z^2) \subset K^3$

$$\rightsquigarrow K(V) = \left\{ \frac{a}{b} \;\middle|\; a, b \in K[X,Y,Z]/(XY-Z^2),\ b \neq 0 \right\}$$

$$= \left\{ \frac{a}{b} \;\middle|\; \begin{array}{l} a, b \text{ regular map on } V, \\ b \text{ not everywhere } 0 \text{ on } V \end{array} \right\}$$

Note that $\frac{X}{Z} = \frac{Z}{Y}$ in $K(V)$
because $XY = Z^2$ in $\Gamma(V)$.

<u>Def</u> A rational function $f \in K(V)$ is <u>defined at</u> $P \in V$ if $f = \frac{a}{b}$ for some $a, b \in \Gamma(V)$ with $b(P) \neq 0$.
We then write $f(P) = \frac{a(P)}{b(P)} \in K$.

Rmk If $f = \frac{a}{b}$ for some a, b with

$a(P) \neq 0$ and $b(P) = 0$, then f is not defined at P.

Pf Assume $\frac{a}{b} = \frac{a'}{b'}$ with $b'(P) \neq 0$.

$$\Rightarrow \underbrace{a(P)}_{\neq 0}\underbrace{b'(P)}_{\neq 0} = a'(P)\underbrace{b(P)}_{=0} \quad ↯ \qquad \square$$

Ex A $f = \frac{x}{z} = \frac{z}{y}$ is defined at all points $(x,y,z) \in V$ with $z \neq 0$ or $y \neq 0$.

It is not defined at $(x,0,0)$ for any $x \neq 0$.

Lemma 2.51 The set U_f of points $P \in V$ at which $f \in K(V)$ is defined is a nonempty open subset of V (open w.r.t. the subspace topology on V), i.e. it's the intersection of an open subset of K^n with V.

Rmk Equivalently: The set of points $P \in V$ at which f isn't defined is closed (= algebraic).

Pf If $f = \frac{a}{b}$ with $b \in \Gamma(V)$ not everywhere 0 on V.

$\Rightarrow$ f is defined (at least) at every point $Q \in V$ with $Q \notin V(b)$.

$\emptyset \neq V \setminus V(b)$ is an open subset of V.

For any $P \in U_f$, we can find a, b as above with $b(P) \neq 0$, so $P \in V \setminus V(b)$.

$\Rightarrow$ U_f is covered by open sets.

$\Rightarrow$ U is open. $\square$

Ese A We know that f is not defined at any point in $\{(x,0,0) \mid x \neq 0\}$.

$\Rightarrow$ f is not defined at any point in the Zariski closure $\{(x,0,0) \mid x \in K\}$.

$\Rightarrow$ f is not defined at $(0,0,0)$.

For $K = \mathbb{C}$, for example, the limit $f(P)$ as $P \to (0,0,0)$ can depend on the path!

$$\underbrace{P = (t,t,t)}_{t \to \infty\ (0,0,0)} \in V \rightsquigarrow f(P) = \frac{t}{t} = 1 \xrightarrow{t \to 0} 1$$

$\neq$

$$\underbrace{P = (t^{3/2}, t^{1/2}, t)}_{t \to 0\ (0,0,0)} \in V \rightsquigarrow f(P) = \frac{t^{3/2}}{t^{1/2}} = t \xrightarrow{t \to 0} 0$$

$\Rightarrow$ no continuous ext. to $(0,0,0)$