2.11 Morphisms

<u>Def</u> Let $V \subseteq K^n$ and $W \subseteq K^m$ be algebraic subsets. A <u>morphism</u> (= <u>regular map</u> = <u>polynomial map</u>) $\varphi : V \to W$ is a map $V \to W$ which is given by polynomials: There exist $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ such that
$$\varphi(P) = (f_1(P), \dots, f_m(P)) \in W \quad \forall P \in V.$$

<u>Ex</u>
$$f : K \longrightarrow V(Y - X^2) \subseteq K^2$$
$$x \longmapsto (x, x^2)$$

<u>Ex</u> The identity $id : V \to V$

<u>Ex</u> An inclusion $V \to W$, where $V \subseteq W \subseteq K^n$.

<u>Rmk</u> If $\varphi : A \to B$ and $\psi : B \to C$ are morphisms, then the composition $\psi \circ \varphi : A \to C$ is a morphism.

<u>Rmk</u> Morphisms $\varphi : \underset{\substack{\cap \\ K^n}}{V} \longrightarrow K^m$ correspond exactly to tuples $(f_1, \dots, f_m)$ of functions $f_i \in \Gamma(V)$ $\quad (f_i : V \to K)$.

In particular, morphisms $\varphi : V \to K$ correspond exactly to elements of $\Gamma(V)$.

How to tell whether the image of $\varphi: V \to K^m$ is contained in W?

<u>Lemma 2.38</u> $\varphi(V) \subseteq W$ if and only if

$$h(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)) \in I(V)$$

for all $h(y_1,\dots,y_m) \in I(W)$.

<u>Pf</u> $\varphi(P) \in W = V(I(W)) \quad \forall P \in V$

$$\iff h(\underbrace{\varphi(P)}) = 0 \quad \forall h \in I(W) \quad \forall P \in V$$

$$\quad\quad \| $$

$$h(f_1(P),\dots,f_m(P))$$

$$\iff h(f_1,\dots,f_m) \in I(V) \quad \forall h \in I(W). \quad \square$$

<u>Def</u> For any morphism $\varphi: V \to W$, we consider the K-algebra homomorphism

$$\varphi^*: \Gamma(W) \longrightarrow \Gamma(V)$$
$$f \longmapsto f \circ \varphi$$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, dashed, "f\circ\varphi"'] & W \arrow[d, "f"] \\ & K \end{tikzcd}$$

φ^* is called the <u>pullback function</u> of φ.

(Sometimes, φ^* is instead denoted by $\tilde{\varphi}$.)

Thm 2.40 We get a bijection

$$\left(\text{morphisms } V \to W\right) \longleftrightarrow \left(\begin{array}{c}K\text{-algebra homomorphisms}\\ \Gamma(W) \to \Gamma(V)\end{array}\right)$$

$$\varphi \longleftrightarrow \varphi^*$$

Pf How to determine φ from φ^*?

Let $y_i \in \Gamma(W)$ be the function mapping any point in $\underset{K^m}{\overset{}{\underset{\cap}{W}}}$ to its i-th coordinate.

Then, $\varphi(P) = (y_1(\varphi(P)), \dots, y_m(\varphi(P)))$

$$= (\varphi^*(y_1)(P), \dots, \varphi^*(y_m)(P))$$

$\boxed{\text{The } m \text{ elements of } \Gamma(V) \text{ defining the morphism } \varphi: V \to W.}$

$\square$

Rmks a) $id: V \to V$ corr. to $id: \Gamma(V) \to \Gamma(V)$

b) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[dr, dashed] & B \arrow[r, "\psi"] \arrow[d, dashed] & C \arrow[dl, "f"] \\
& K &
\end{tikzcd}$$

Rmk Hence,

$$\left\{\begin{matrix}\text{alg. subsets of } K^n \\ \text{for some } n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finitely generated} \\ \text{reduced } K\text{-algebras}\end{matrix}\right\}$$

$$V \longmapsto \Gamma(V) = K[x_1, \dots, x_n] / I(V)$$

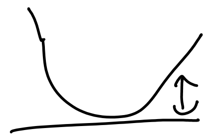
$$\varphi \longmapsto \varphi^*$$

is a contravariant equivalence of categories.

Def A morphism $\varphi : V \to W$ is an isomorphism if it has an inverse morphism $\psi : W \to V$. (with $\varphi \circ \psi = id_W$, $\psi \circ \varphi = id_V$).

Rmk $\varphi : V \to W$ is an isomorphism if and only if $\varphi^* : \Gamma(W) \to \Gamma(V)$ is.

Ex The inverse of
$$K \longrightarrow V(Y - X^2)$$
$$x \longmapsto (x, x^2)$$
is
$$x \longleftarrow\!\shortmid (x, y).$$



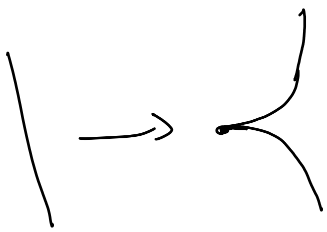
Ex Any translation in K^n is a isomorphism.

Ex Any invertible linear map $K^n \to K^n$ is an isomorphism.

__Warning__ Not every bijective morphism $f: V \to W$ is an isomorphism! (Just like not every bijective continuous map is a homeomorphism.)

__Ex__ $\varphi: K \longrightarrow V(X^3 - Y^2) \subseteq K^2$

$t \longmapsto (t^2, t^3)$



is a bijection with inverse

$\frac{Y}{X} \longleftarrow\!\!| \; (X, Y)$

because $\varphi^*: K[X,Y]/(X^3-Y^2) \longrightarrow K[T]$

$$X \longmapsto T^2$$
$$Y \longmapsto T^3$$

is not an isomorphism because T does not lie in the image.

__Thm 2.41__ Any morphism $\varphi: \underset{K^n}{\overset{\cap}{A}} \to \underset{K^m}{\overset{\cap}{B}}$ is continuous (w.r.t. the Zariski topologies on A and B).

__Pf__ $\varphi^{-1}(\underbrace{V(J)}_{\text{closed subset of } K^m}) = \{P \in A \mid \underbrace{\varphi(P) \in V(J)}\} = \underbrace{V(\varphi^*(J))}_{\text{closed subset of } A}$

$\iff f(\varphi(P)) = 0 \;\forall f \in J$

$\iff \varphi^*(f)(P) = 0 \;\forall f \in J$

$\iff P \in V(\varphi^*(J))$

$\square$

Thm 2.42 $\varphi^*: \Gamma(W) \longrightarrow \Gamma(V)$ is injective if and only if $\varphi(V)$ is (Zariski) dense in W.

Def Such a morphism φ is called dominant.

Pf Let $V \subseteq K^n$, $W \subseteq K^m$.

φ^* injective

$$\Longleftrightarrow \forall f \in \Gamma(W) \text{ if } \underbrace{\underset{\substack{\| \\ f\circ\varphi}}{\varphi^*(f)} = 0 \text{ on } V}_{f = 0 \text{ on } \varphi(V)}, \text{ then } f = 0 \text{ on } W$$

$$\Longleftrightarrow \forall f \in K[Y_1, \dots, Y_m] \text{ if } \underbrace{f = 0 \text{ on } \varphi(V)}_{\Leftrightarrow f \in I(\varphi(V))}, \text{ then } \underbrace{f = 0 \text{ on } W}_{f \in I(W)}.$$

$$\Longleftrightarrow I(\varphi(V)) \subseteq I(W)$$

$$\cancel{\Longleftrightarrow \varphi(V) \supseteq W}$$

$$\Longrightarrow \underset{\substack{\| \\ \overline{\varphi(V)}}}{V(I(\varphi(V)))} \supseteq \underset{\substack{\| \\ W}}{V(I(W))}$$

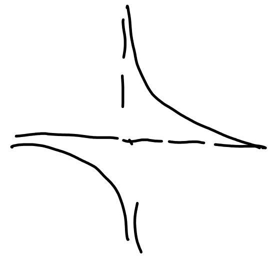
closure of $\varphi(V) \subseteq K^n$ w.r.t. the Zariski topology.

$$\boxed{A \supseteq B \Longrightarrow I(A) \subseteq I(B) \quad \not\Longleftarrow}$$

To do: show "$\Leftarrow$" $\square$

Exe $\varphi: V(XY-1) \longrightarrow K$

$\qquad (x,y) \longmapsto x$



has image $K \setminus \{0\}$ (dense in K).

$$\varphi^*: K[T] \longrightarrow K[X,Y]/(XY-1) \cong K[X, \tfrac{1}{X}]$$

$$T \longmapsto X$$

is injective.

Rmk The composition of two dominant morphisms is dominant.

Thm 2.43 $\varphi^*: \Gamma(W) \to \Gamma(V)$ is surjective if and only if $\varphi: V \to \varphi(V)$ is an isomorphism (onto its image).

Pf Let $I \subseteq \Gamma(W)$ be the kernel of φ. Since $\Gamma(V)$ is reduced, the ideal I of $\Gamma(W)$ is a radical ideal. Let $\underset{V(I)}{\underset{\shortparallel}{W'}} \subseteq W$ be the corresponding algebraic subset of W.

We get a map $\varphi^*: \underset{\Gamma(W')}{\underset{\shortparallel}{\Gamma(W)/I}} \longrightarrow \Gamma(V)$

corr. to $\varphi: V \longrightarrow W'$ (the closure of the image of $\varphi: V \to W$ is W' by the previous theorem).

Then, $\varphi^* : \Gamma(W) \longrightarrow \Gamma(V)$ is surjective if and only if $\varphi^* : \Gamma(W') = \Gamma(W)/I \longrightarrow \Gamma(V)$ is an isomorphism. In other words:

$\varphi : V \longrightarrow W'$ is an isomorphism. $\square$