c) $Y = \bigcup_{i=1}^{n} \underbrace{(X_i \cap Y)}_{\text{alg.}}$

$\Rightarrow$ $Y = X_i \cap Y$ for some i

$\uparrow$

Y irred

$\Rightarrow$ $Y \subseteq X_i$ for some i

b) Let $X = X_1 \cup \dots \cup X_n = Y_1 \cup \dots \cup Y_m$ as above.

Say X_i doesn't occur among $Y_1, \dots, Y_m$.

$\left.\begin{array}{l}\text{By c), } X_i \subseteq Y_j \text{ for some } j. \\ \text{By c), } Y_j \subseteq X_k \text{ for some } k.\end{array}\right\} X_i \subseteq Y_j \subseteq X_k$

Since $X_i \not\subseteq X_k$ for $i \neq k$, it follows that we have $i = k$ and therefore equality:

$X_i = Y_j$. □

<u>Summary</u>

We have bijections

$$\begin{array}{ccc}
(\text{alg. subsets of } K^n) & \longleftrightarrow & (\text{radical ideals of } K[x_1, \dots, x_n]) \\
\cup\!| & & \cup\!| \\
(\text{irreducible alg. subsets of } K^n) & \longleftrightarrow & (\text{prime ideals of } K[x_1, \dots, x_n]) \\
\cup\!| & & \cup\!| \\
(\text{points in } K^n) & \longleftrightarrow & (\text{maximal ideals of } K[x_1, \dots, x_n])
\end{array}$$

2.10. Coordinate rings

Def The coordinate ring of an algebraic subset V of K^n is $\Gamma(V) := K[X_1, \dots, X_n]/I(V)$.

Rmk a) $\Gamma(V)$ is a reduced ring: for any $f \in \Gamma(V)$: if $f^n = 0$ for some $n \geq 1$, then $f = 0$.

b) V is irreducible if and only if $\Gamma(V)$ is an integral domain: if $fg = 0$, then f or $g = 0$.

c) $|V| = 1$ if and only if $\Gamma(V)$ is a field.

Thm 2.31 $\Gamma(V)$ is the ring of functions $f : V \longrightarrow K$ given by some polynomial $g \in K[X_1, \dots, X_n]$: $f = g|_V$

Pf Two polynomials $g_1, g_2 \in K[X_1, \dots, X_n]$ agree on V if and only if $g_1 - g_2$ vanishes everywhere on V, i.e. $g_1 - g_2 \in I(V)$. $\square$

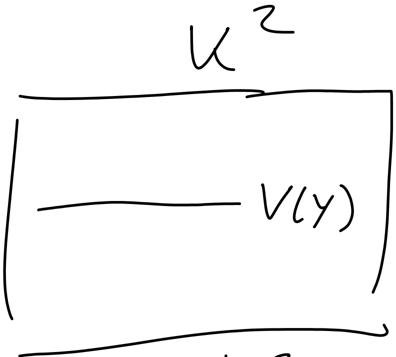
Ex The function X_i sending any point to its i-th coordinate.

Rmk If $V \subseteq W$ we get a surjective ring homomorphism

$$\Gamma(W) \longrightarrow \Gamma(V).$$
$$f \longmapsto f|_V$$

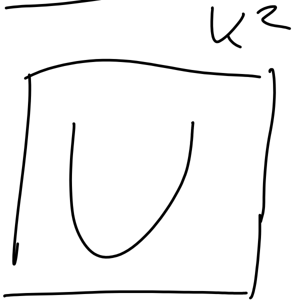
Ex $\Gamma(K^n) = K[X_1, \dots, X_n]/0 = K[X_1, \dots, X_n]$

Ex



$$\Gamma(\underbrace{V(Y)}_{\substack{X\text{-axis} \\ \text{in } K^2}}) = K[X,Y]/(Y) \cong K[X]$$
$$X \mapsto X$$
$$Y \mapsto 0$$

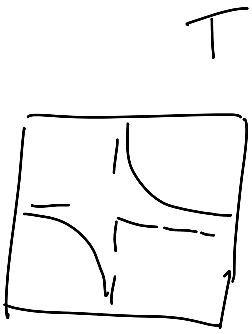
Ex



$$\Gamma(V(Y-X^2)) = K[X,Y]/(Y-X^2) \cong K[X]$$
$$Y \mapsto X^2$$
$$X \mapsto X$$

$$\left[\begin{array}{ll} R[T]/(T-r) \cong R & \\ & \text{for any } r \in R \\ T \mapsto r & \end{array} \right]$$

Ex



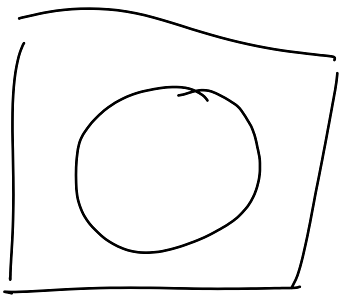
$$\Gamma(V(XY-1)) = K[X,Y]/(XY-1) \cong K[X, \tfrac{1}{X}]$$
$$Y \mapsto \frac{1}{X}$$

$$K[X, \tfrac{1}{X}] \ni 1 + 2X^3 \cdot \left(\tfrac{1}{X}\right)^2 + 3X \cdot \left(\tfrac{1}{X}\right)^4 + \dots$$

Warning $\underbrace{K(X)}_{\text{field of rational functions}} \ni \frac{1}{X+1} \notin \underbrace{K[X, \tfrac{1}{X}]}_{\text{ring of Laurent polynomials}}.$

Ex Assume $K = \mathbb{C}$ (or at least $\operatorname{char}(K) \neq 2$).

Then, $\Gamma(V(X^2+Y^2-1)) = K[X,Y]/(X^2+Y^2-1)$



$$\cong K[X, \sqrt{1-X^2}] \qquad Y \mapsto \sqrt{1-X^2}$$

Fundamental principle

You can determine all "intrinsic" properties of an algebraic subset V of K^n from its coordinate ring.

Rmk There are bijections

$$\begin{array}{ccc} (\text{algebraic subsets } W \subseteq V \subseteq K^n) & \leftrightarrow & (\text{radical ideals of } \Gamma(V)) \\ \cup\!| & & \cup\!| \\ (\text{irred. alg. subsets } W \subseteq V) & \leftrightarrow & (\text{prime ideals of } \Gamma(V)) \\ \cup\!| & & \cup\!| \\ (\text{points on } V) & \leftrightarrow & (\text{maximal ideals of } \Gamma(V)) \end{array}$$

Pf Use the bijections

$$(\text{ideals } I \subseteq J \subseteq R) \leftrightarrow (\text{ideals } J' \text{ of } R/I)$$

for fixed I. (J = preimage of J' under the quotient map $R \twoheadrightarrow R/I$). $\square$

Chinese remainder theorem

Let $I_1, \dots, I_m$ be ideals of a ring R. If they are pairwise coprime ($I_i + I_j = R$ for all $i \neq j$), then we have a ring isomorphism

$$R/_{I_1 \cap \dots \cap I_m} \cong R/_{I_1} \times \dots \times R/_{I_m}.$$

$$r \bmod I_1 \cap \dots \cap I_m \mapsto (r \bmod I_1, \dots, r \bmod I_m).$$

Furthermore, $I_1 \cdots I_m = I_1 \cap \dots \cap I_m$.

Cor 2.32 Let $V_1, \dots, V_m$ be algebraic subsets of K^n. If they are pairwise disjoint, then

$$\Gamma(V_1 \cup \dots \cup V_m) \cong \Gamma(V_1) \times \dots \times \Gamma(V_m)$$

$$f \mapsto (f|_{V_1}, \dots, f|_{V_m})$$

Pf Let $I_i = I(V_i)$.

$\Rightarrow V_1 \cup \dots \cup V_m = V(I_1 \cap \dots \cap I_m)$

Each I_i is a radical ideal.

$\Rightarrow I_1 \cap \dots \cap I_m$ is a radical ideal

$\Rightarrow I(V_1 \cup \dots \cup V_m) = I_1 \cap \dots \cap I_m$.

$\Rightarrow \Gamma(V_1 \cup \dots \cup V_m) = K[x_1, \dots, x_n]/(I_1 \cap \dots \cap I_m)$

$\Gamma(V_i) = K[x_1, \dots, x_n]/I_i$.

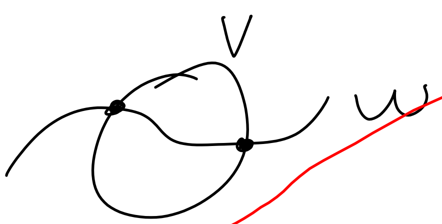
Apply the Chinese remainder theorem. $\square$

/ O

More generally:

Thm 2.33 Let V, W be algebraic subsets of K^n. Then,

$$\Gamma(V \cup W) \cong \{(f,g) \in \Gamma(V) \times \Gamma(W) \mid f|_{V \cap W} = g|_{V \cap W}\}$$
$$h \longmapsto (h|_V, h|_W)$$



Pf HW. "$\square$"

You can determine the number of points in V from $\Gamma(V)$:

Lemma 2.34 Let $V \subseteq K^n$ be a finite set consisting of m points. Then

$$\Gamma(V) = \underbrace{K \times \cdots \times K}_{m \text{ times}}.$$

In particular, the dimension of $\Gamma(V)$ as a K-vector space is $\dim_K(\Gamma(V)) = m$.

Pf If $m=1$: The ring of functions
$\underset{\{p\}}{\overset{\shortparallel}{V}} \longrightarrow K$ is K. (Any such function is given by a constant polynomial.)

For $m>1$, apply Cor 2.32. $\square$

<u>Cor 2.35</u> An algebraic subset $V \subseteq K^n$ is finite if and only if $\dim_K(\Gamma(V)) < \infty$.

Pf "$\Rightarrow$" clear

"$\Leftarrow$" If V contains at least m points $P_1, \ldots, P_m$, we have a surjection

$$\Gamma(V) \longrightarrow \Gamma(\{P_1, \ldots, P_m\}).$$

$$\dim_K(\Gamma(V)) \geq \dim_K(\Gamma(\{P_1, \ldots, P_m\})) = m. \quad \square$$

Rmk This is equivalent to $\Gamma(V)$ being an integral (= algebraic) K-algebra.

More generally:

<u>Thm 2.36</u> Let I be any ideal of $K[X_1, \ldots, X_n]$. Then, $V(I)$ is finite if and only if $\dim_K(K[X_1, \ldots, X_n]/I) < \infty$.

In that case, $|V(I)| \leq \dim_K(K[X_1, \ldots, X_n]/I)$.

Ex $I = (x(x-1)^2(x-2)^5)$
$= (x^8 + \dots + 0)$
$V(I) = \{0, 1, 2\}$

$K[x]/I$ has K-basis $1, x, \dots, x^7$

Always, $\#\{\text{roots of } f(x)\} \leq \deg(f)$.

$$\underbrace{\#\{\text{roots of } f(x)\}}_{= V(f)} \leq \underbrace{\deg(f)}_{= \dim_K (K[x]/(f))}$$

This follows from:

Lemma 2.37 Let I be an ideal of a ring extension S of a ring R. Then, S/I is an integral R-algebra if and only if $S/\sqrt{I}$ is an integral R-algebra.

Idea of pf If $\alpha \in S$, $f \in R[x]$ monic, $f(\alpha) = 0$ in $S/\sqrt{I}$ (so $f(\alpha) \in \sqrt{I}$).
$\Rightarrow f(\alpha)^n \in I$ for some n
$\Rightarrow f(\alpha)^n = 0$ in S/I, $f^n \in R[x]$ monic. $\square$

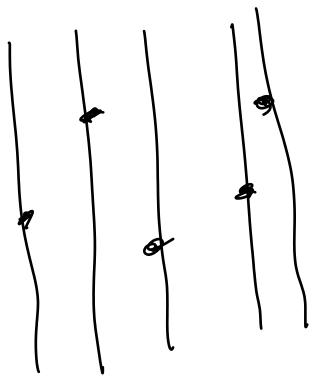
Direct pf of Thm 2.36

$$\sqrt{I} \supseteq I, \text{ so } \dim_K\big(\underbrace{K[X_1,\dots,X_n]/\sqrt{I}}_{\Gamma(V(I))}\big)$$

$$\le \dim_K\big(K[X_1,\dots,X_n]/I\big)$$

Assume $V(I)=\{P_1,\dots,P_m\}$ with $P_j=(a_{1j},\dots,a_{nj})$.

$$\Rightarrow (X_i-a_{i1})\cdots(X_i-a_{im}) \in I(V(I))=\sqrt{I}$$



$$\Rightarrow \exists N \ge 1\ \forall i: \big((X_i-a_{i1})\cdots(X_i-a_{im})\big)^N \in I$$

$$\Rightarrow \dim_K(\dots/I) \le \dim_K\Big(\dots/\underbrace{\big(\text{id. gen. by } ((X_i-a_{i1})\cdots(X_i-a_{im}))^N\big)}_{T}\Big)$$

$$= (mN)^n.$$

basis: $\{X_1^{e_1}\cdots X_n^{e_n} \mid 0 \le e_1,\dots,e_n < Nm\}$ of T

$\square$