Algebraic Geometry

1. Overview

Let K be a field.

An <u>algebraic subset</u> of K^n is the set of solutions $(x_1, \dots, x_n) \in K^n$ to a system of polynomial equations:

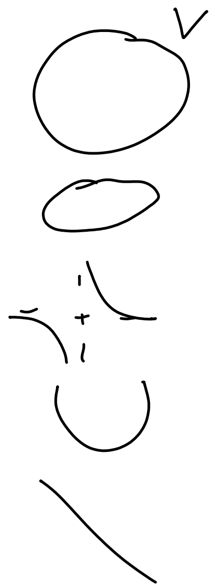
$$f_1(x_1, \dots, x_n) = 0, \quad f_1 \in K[x_1, \dots, x_n]$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$f_m(x_1, \dots, x_n) = 0, \quad f_m \in K[x_1, \dots, x_n]$$

<u>Ex</u>

Conic (= conic section):

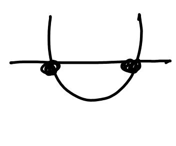
- Circle $V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
- Ellipse $\quad 2x^2 + 3y^2 = 1$
- Hyperbola $\quad xy = 1$
- Parabola $\quad y = x^2$

Line $\quad x + 2y = 3$



Point $\{(1,2)\} = \{(x,y) \in \mathbb{R}^2 \mid y = 2x,\ x + 1 = y\}$
$= \{ \quad \mid x = 1,\ y = 2\}$

Two points $\{(0,0), (1,0)\} = \{ \quad \mid x(x-1) = y,\ y = 0\}$



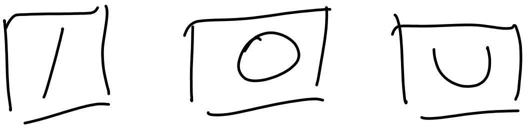
$\vdots$

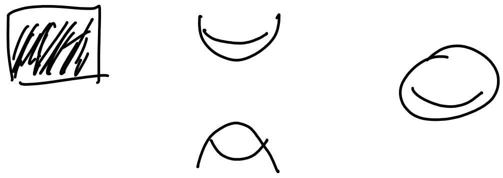
Questions

- Is V a set of just finitely many points?

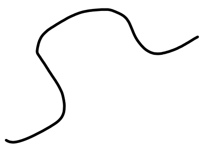
  If so, how many?
- What is the "dimension" of V?

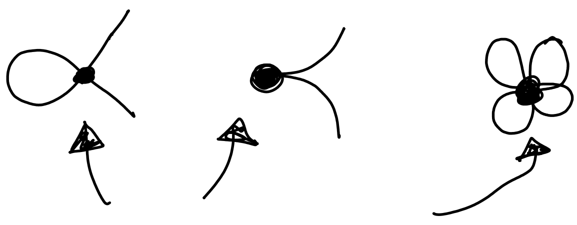
  dim = 0: 

  dim = 1: 

  dim = 2: 

- Is V "smooth"?

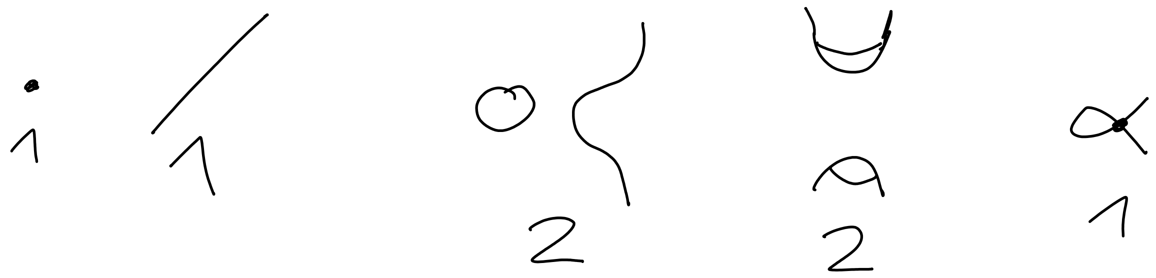
  smooth 

  not smooth 

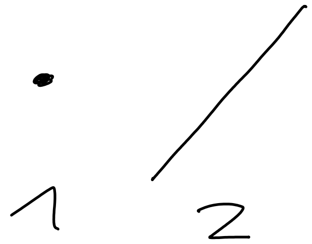
- If not, what are the "singularities" look like?

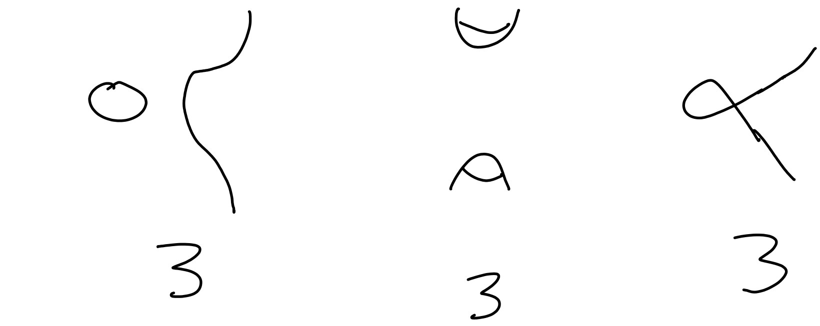
Real algebraic geometry ($K = \mathbb{R}$)

- How many connected components does V have?



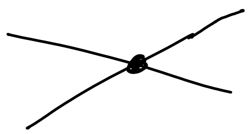
- How many connected components does the complement $\mathbb{R}^n \setminus V$ have?

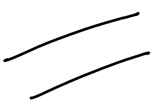




Intersection theory

- In how many points do two lines $L_1 \neq L_2 \subseteq K^2$ intersect?

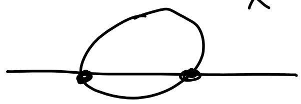
Usually 1 

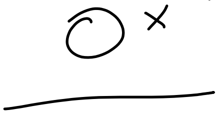
Occasionally 0  (if L_1, L_2 are parallel)

$\rightsquigarrow$ Always 1 in the projective plane.

- In how many points does a line intersect a conic?

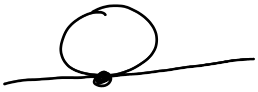
$x^2+(y-2)^2=100$

Sometimes 2  $y=1$

Sometimes 0  $x^2+(y-2)^2=100$

$y=-1000$

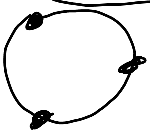
(can't happen in algebraically closed fields like $\mathbb{C}$)

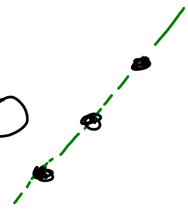
Occasionally 1  (with "multiplicity" 2)

Enumerative geometry

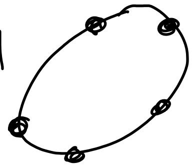
- How many circles are there through three given points p_1, p_2, p_3?

(distinct)

Usually 1 

Occasionally 0 

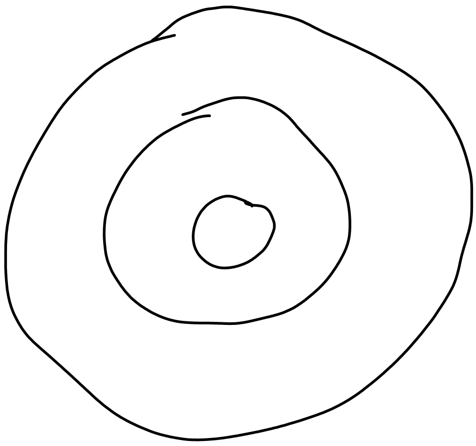
- How many conics are there through five given points p_1, p_2, p_3, p_4, p_5?

Usually 1 

- How many circles are there tangent to three given circles?

Sometimes 8

Sometimes 0



1

2

3

4

5

6

But 7 is impossible!

- How many lines are there that intersect four given lines in three-dimensional space?

  Usually 2

- How many lines are there on a given cubic surface (surface defined by a pol. of degree 3)?

  Usually 27 (if $K = \mathbb{C}$).

Prerequisites

Algebra: rings, modules, fields, ...
algebraically closed fields

References

Fulton
Brooke Ullery's lecture notes

Grade

70% weekly homeworks
(dropping the two lowest scores)

30% take-home exam

2. Affine varieties

2.1. Algebraic sets

Let K be a field.

Rmk In algebraic geometry, the set of points in K^n is often also denoted by $\mathbb{A}^n$ or $\mathbb{A}^n_K$ and called the n-dimensional affine space (over K).

Def For a set $S \subseteq K[x_1, \dots, x_n]$ of polynomials, we denote by

$$V(S) = \{ P \in K^n \mid f(P) = 0 \ \forall f \in S \}$$

the corresponding set of zeros.

Rmk If $S \subseteq S'$, then $V(S) \supseteq V(S')$.
(V is inclusion-reversing.)

Def A subset $X \subseteq K^n$ is algebraic if

$$X = V(S) \text{ for some } S \subseteq K[x_1, \dots, x_n].$$

Rmk This differs from the definition in chapter 1, where we only allowed finitely many polynomial equations.

We'll soon (in chapter 2.2) see

that the two definitions are equivalent!

<u>Ex</u> $V(\{x_2 - x_1^2\}) = \{(x_1, x_2) \in K^2 / x_2 = x_1^2\}$

if $n=2$.

<u>Ex</u> $V(\{x_1 - a_1, \dots, x_n - a_n\}) = \{(a_1, \dots, a_n)\}$,

so every one-point subset $\{P\} \subseteq K^n$ is algebraic.

<u>Note</u>
- If $f(P) = 0$, then $f(P) \cdot g(P) = 0 \ \forall g \in K[x_1, \dots, x_n]$
- If $f(P) = 0$ and $g(P) = 0$, then $f(P) + g(P) = 0$.

<u>Cor 2.1</u> If I is the ideal of $K[x_1, \dots, x_n]$ generated by S,

then $V(I) = V(S)$.

<u>Pf</u> "$\subseteq$" follows from $I \supseteq S$

"$\supseteq$" Every element of I can be written as

$$\underbrace{\underbrace{f_1}_{0 \text{ at } P} g_1 + \dots + \underbrace{f_r}_{0 \text{ at } P} g_r}_{0 \text{ at } P} \text{ with } f_1, \dots, f_r \in S, \ g_1, \dots, g_r \in K[x_1, \dots, x_n]$$

for all $P \in S$ $\square$

Lemma 2.2

a) For any collection of ideals I_α,

$$\bigcap_\alpha V(I_\alpha) = V\left(\bigcup_\alpha I_\alpha\right)$$
$$= V(\text{ideal generated by } \textstyle\bigcup_\alpha I_\alpha).$$

b) For any two ideals I, J

$$V(I) \cup V(J) = V(\underbrace{I \cdot J})$$

ideal generated by polynomials of the form $f \cdot g$ with $f \in I$, $g \in J$.

c) $V(0) = K^n$

d) $V(1) = \emptyset$.

Pf a) clear

b) $P \in LHS \iff P \in V(I)$ or $P \in V(J)$

$\iff \forall f \in I: f(P) = 0$ or $\forall g \in J: g(P) = 0$

$\iff \forall f \in I, g \in J: f(P) = 0$ or $g(P) = 0$

$\iff \forall f \in I, g \in J: f(P) \cdot g(P) = 0$

$\iff P \in RHS$

c) $P \in V(0) \iff 0 = 0$ d) $P \in V(1) \iff 1 = 0$ $\square$