Math 286X: Arithmetic Statistics

Spring 2020

Problem set #5

Problem 1. Consider a measure-preserving action of a countable group G on a set X. Let $\widetilde{\mathcal{F}}$ be a measurable almost fundamental domain for this action, with volume $0 < \mathrm{vol}(\widetilde{\mathcal{F}}) < \infty$. Show that the corresponding fundamental domain $\mathcal{F}$ also has volume $0 < \mathrm{vol}(\mathcal{F}) < \infty$.

Solution. The characteristic function $\chi_{\mathcal{F}}$ satisfies $0 < \chi_{\mathcal{F}}(x) \leqslant 1$ for any $x \in \widetilde{\mathcal{F}}$. This immediately implies the claim.

Step by step: Let A_n be the set of elements x of $\widetilde{\mathcal{F}}$ such that there are exactly n distinct $g \in G$ with $gx \in \widetilde{\mathcal{F}}$. We have seen in class that A_n is measurable. By definition, $\widetilde{\mathcal{F}} = \bigsqcup_{n=1}^{\infty} A_n$ and $\mathcal{F} = \bigsqcup_{n=1}^{\infty} A_n^{\sqcup \frac{1}{n}}$. Hence, $\mathrm{vol}(\widetilde{\mathcal{F}}) = \sum_{n=1}^{\infty} \mathrm{vol}(A_n)$ and $\mathrm{vol}(\mathcal{F}) = \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{vol}(A_n)$. In particular, $\mathrm{vol}(\mathcal{F}) \leqslant \mathrm{vol}(\widetilde{\mathcal{F}})$. On the other hand, $\mathrm{vol}(\widetilde{\mathcal{F}}) > 0$ implies that $\mathrm{vol}(A_n) > 0$ for some n, so $\mathrm{vol}(\mathcal{F}) > 0$. ☐

Problem 2. Order the full integer lattices $\Lambda \subseteq \mathbb{Z}^n$ by their covolume.

a) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. Show that

$$\mathbb{P}(e_1 \in \Lambda \mid \Lambda \subseteq \mathbb{Z}^n \text{ full lattice}) = 0.$$

Solution. Consider the embedding $\varphi : \mathbb{Z}^{n-1} \hookrightarrow \mathbb{Z}^n$ sending $(x_2, \ldots, x_n)$ to $(0, x_2, \ldots, x_n)$. We obtain a covolume-preserving bijection

$$\{\text{full lattice } \Lambda' \subseteq \mathbb{Z}^{n-1}\} \longleftrightarrow \{\text{full lattice } \Lambda \subseteq \mathbb{Z}^n \text{ containing } e_1\}$$

sending Λ' to $e_1\mathbb{Z} + \varphi(\Lambda')$. We have seen in class that there are $\asymp T^{n-1}$ full lattices $\Lambda' \subseteq \mathbb{Z}^{n-1}$ and that there are $\asymp T^n$ full lattices $\Lambda \subseteq \mathbb{Z}^n$ with covolume at most T. ☐

b) Let $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ be the projection onto the first $n-1$ coordinates. Then, $\pi(\Lambda) \subseteq \mathbb{Z}^{n-1}$ is always a full lattice. Show that

$$\mathbb{P}(\pi(\Lambda) = \mathbb{Z}^{n-1} \mid \Lambda \subseteq \mathbb{Z}^n \text{ full lattice}) = \frac{1}{\zeta(2) \cdots \zeta(n)}.$$

Solution. A lattice Λ satisfies $\pi(\Lambda) = \mathbb{Z}^{n-1}$ if and only if its Hermite normal form looks like the identity matrix in the first $n-1$ columns. The number of such matrices in Hermite normal form of determinant at most T is $\sum_{a \leqslant T} a^{n-1} \sim \frac{1}{n}T^n$. The claim follows, since we have shown that the total number of lattices of covolume at most T is $\sum_{a_1 \cdots a_n \leqslant T} \prod_{i=1}^n a_i^{i-1} \sim \frac{1}{n}\zeta(2)\cdots\zeta(n)T^n$. □

Problem 3 (Mahler's criterion)**.** Equip $\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ with the quotient topology. Let X be a closed subset of $\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$. Show that X is compact if and only if there exist $0 < C \leqslant C' < \infty$ such that the successive minima $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ of any lattice Λ corresponding to an element of X satisfy $C \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant C'$.
Hint: Use the Iwasawa decomposition and Siegel's almost fundamental domain.

Solution. Use the Iwasawa decomposition $\mathrm{GL}_n(\mathbb{R}) = NAK$ and the fact that any $\mathrm{GL}_n(\mathbb{Z})$-orbit contains an element in Siegel's almost fundamental domain $N'A'K$. Let $\pi : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ be the (continuous) projection map. Recall that if $g = nak$ with $n \in N'$, $a \in A'$, $k \in K$ corresponds to the lattice Λ with successive minima $\lambda_1 \leqslant \cdots \leqslant \lambda_n$, then the diagonal entries of a satisfy $a_i \asymp \lambda_i$. The same holds true if we replace N' by a slightly larger open set N'' (the set of matrices in N whose entries are all smaller than 1, say).

"⇒" Let us cover A' by open sets $R_{D,D'}$ consisting of diagonal matrices $a \in A'$ with $D < a_1, \ldots, a_n < D'$. We obtain an open cover $\pi(N''R_{D,D'}K)$ of $\mathrm{GL}_N(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$. If X is compact, then it is contained in $\pi(N''R_{D,D'}K)$ for some fixed D, D'. Since $a_i \asymp \lambda_i$, this implies $C \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant C'$ for some C, C' and all lattices in X.

"⇐" The subset $S_{D,D'}$ of A' consisting of diagonal matrices $a \in A'$ with $D \leqslant a_1, \ldots, a_n \leqslant D'$ is compact for any $D, D' > 0$. Hence, so is the image $\pi(N'S_{D,D'}K)$. For sufficiently small D and large D', it contains the closed set X, which must therefore also be compact. □

Problem 4 (lattice points in cusps)**.** For any $\alpha \in \mathbb{R}$ and any $X > 0$, consider the compact set

$$S_\alpha(X) = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x \leqslant X \text{ and } |y - \alpha x| \leqslant \frac{1}{x}\}$$

and let $N_\alpha(X) = \#(S_\alpha(X) \cap \mathbb{Z}^2)$.

a) Show that its Lebesgue measure is $\mathrm{vol}(S_\alpha(X)) = 2\log X$.

Solution. The volume is

$$\int_1^X \frac{2}{x}\mathrm{d}x = 2\log X. \qquad \square$$

b) (strangeness) Let $\alpha = \frac{p}{q}$ with $\gcd(p,q) = 1$. Show that

$$N_\alpha(X) = \frac{X}{q} + \mathcal{O}(q).$$

Solution. For any $1 \leqslant x \leqslant X$ and $y \in \mathbb{Z}$, we have $(x,y) \in S_\alpha(X)$ if and only if y lies in the interval $[\alpha x - \frac{1}{x}, \alpha x + \frac{1}{x}]$ of length $\frac{2}{x} \leqslant 2$.

For any fixed $1 \leqslant x \leqslant X$, this interval contains at most 3 integers y.

If $x \equiv 0 \mod q$ and $x > 1$, there is exactly one integer, $y = \frac{px}{q}$ in this interval.

If $x \not\equiv 0 \mod q$ and $x > q$, the interval contains no integers at all: $\frac{px}{q}$ has distance at least $\frac{1}{q}$ from the closest integer.

Therefore,

$$N_\alpha(X) = \sum_{1 \leqslant x \leqslant q} \mathcal{O}(1) + \sum_{\substack{q < x \leqslant X, \\ x \equiv 0 \mod q}} 1 = \frac{X}{q} + \mathcal{O}(q). \qquad \square$$

c) (Dirichlet's approximation theorem) Show that for any $\alpha \in \mathbb{R}$, we have

$$\lim_{X \to \infty} N_\alpha(X) = \infty.$$

Solution. If α is rational, the claim follows from b). Assume α is irrational. Consider the parallelogram

$$R_\alpha(X) = \{(x,y) \in \mathbb{R}^2 \mid |x| \leqslant X \text{ and } |y - \alpha x| \leqslant \frac{1}{X}\}$$

of area 4. By Minkowski's first theorem, it contains a nonzero lattice point $(x,y) \in \mathbb{Z}^2$. If $X > 1$, we cannot have $x = 0$. Reflecting across the origin, we can make $x > 0$, so that $(x,y) \in S_\alpha(X)$. Now, note that for $X \to \infty$, we can make $|y - \alpha x|$ arbitrarily close to zero. Since α is irrational, it cannot be equal to zero, so there must be infinitely many points (x,y) as above. $\square$

d) (averaging) For any $a < b$, show that

$$\frac{1}{b-a}\int_a^b N_\alpha(X)\mathrm{d}\alpha = 2\log X + \mathcal{O}\left(1 + \frac{1}{b-a}\right).$$

Solution. We show this using a change of variable $t = y - \alpha x$:

$$\begin{aligned}
\frac{1}{b-a}\int_a^b N_\alpha(X)\mathrm{d}\alpha &= \sum_{x=1}^{X}\sum_{y\in\mathbb{Z}} \frac{1}{b-a}\int_a^b \chi_{[-1/x,1/x]}(y-\alpha x)\mathrm{d}\alpha \\
&= \sum_{x=1}^{X}\sum_{y\in\mathbb{Z}} \frac{1}{b-a}\int_{\mathbb{R}} \chi_{[a,b]}(\alpha)\chi_{[-1/x,1/x]}(y-\alpha x)\mathrm{d}\alpha \\
&= \sum_{x=1}^{X}\sum_{y\in\mathbb{Z}} \frac{1}{b-a}\int_{\mathbb{R}} \chi_{[a,b]}\left(\frac{y-t}{x}\right)\chi_{[-1/x,1/x]}(t)\frac{\mathrm{d}t}{x} \\
&= \sum_{x=1}^{X}\sum_{y\in\mathbb{Z}} \frac{1}{b-a}\int_{-1/x}^{1/x} \chi_{[a,b]}\left(\frac{y-t}{x}\right)\frac{\mathrm{d}t}{x} \\
&= \sum_{x=1}^{X}\sum_{y\in\mathbb{Z}} \frac{1}{b-a}\int_{-1/x}^{1/x} \chi_{[t+ax,t+bx]}(y)\frac{\mathrm{d}t}{x} \\
&= \sum_{x=1}^{X} \frac{1}{b-a}\int_{-1/x}^{1/x} \#(\mathbb{Z}\cap[t+ax,t+bx])\frac{\mathrm{d}t}{x} \\
&= \sum_{x=1}^{X} \frac{1}{b-a}\int_{-1/x}^{1/x} ((b-a)x+\mathcal{O}(1))\frac{\mathrm{d}t}{x} \\
&= \sum_{x=1}^{\lfloor X\rfloor} \frac{2}{x} + \mathcal{O}\left(\frac{1}{b-a}\right) \\
&= 2\log X + \mathcal{O}\left(1+\frac{1}{b-a}\right). \qquad \square
\end{aligned}$$

Problem 5 (smooth functions are swell)**.** Let A be a weighted set on $\mathbb{R}$ whose characteristic function $\chi_A : \mathbb{R} \to \mathbb{R}^{\geqslant 0}$ is smooth and whose support is bounded. Let $k \geqslant 0$. Show that

$$\#((T\cdot A)\cap\mathbb{Z}) = T\cdot \mathrm{vol}(A) + \mathcal{O}_{A,k}(T^{-k})$$

for $T \to \infty$. (Note that the error term is much better than the error term $\mathcal{O}(1)$ we would get if A were an interval!)
Hint: For example, apply the Poisson summation formula or the Euler–Maclaurin formula (both use integration by parts).

Solution. **Using poisson summation** The Poisson summation formula implies that

$$\begin{aligned}\#((T\cdot A)\cap\mathbb{Z}) &= \sum_{n\in\mathbb{Z}}\chi_{T\cdot A}(n)\\ &= \sum_{n\in\mathbb{Z}}\widehat{\chi_{T\cdot A}}(n).\end{aligned}$$

By definition of Fourier transforms, $\widehat{\chi_{T\cdot A}}(n) = T\cdot\widehat{\chi_A}(Tn)$. Furthermore, $\widehat{\chi_A}(0) = \mathrm{vol}(A)$, so $\widehat{\chi_{T\cdot A}}(0) = T\cdot\mathrm{vol}(A)$. On the other hand, since χ_A is smooth and compactly supported, integration by parts shows that $\widehat{\chi_A}(X) = \mathcal{O}_{A,l}(X^{-l})$ for any $l\geqslant 0$ and $|X|\to\infty$. Therefore, $\widehat{\chi_{T\cdot A}}(n) = \mathcal{O}_{A,l}(T\cdot(Tn)^{-l})$ for $n\neq 0$. Summing over all n proves the result (choosing $l\geqslant\max(2,k+1)$).

Using the Euler–Maclaurin formula Since χ_A is compactly supported, the Euler–Maclaurin formula (which can be proven using integration by parts) shows that for any $k\geqslant 1$,

$$\begin{aligned}\#((T\cdot A)\cap\mathbb{Z}) &= \sum_{n\in\mathbb{Z}}\chi_{T\cdot A}(n)\\ &= \int_{\mathbb{R}}\chi_{T\cdot A}(x)\mathrm{d}x + \mathcal{O}_k\left(\int_{\mathbb{R}}\chi_{T\cdot A}^{(k)}(x)\mathrm{d}x\right)\\ &= \mathrm{vol}(T\cdot A) + \mathcal{O}_k\left(T^{-k}\cdot\int_{\mathbb{R}}\chi_A^{(k)}(x)\mathrm{d}x\right)\\ &= T\cdot\mathrm{vol}(A) + \mathcal{O}_{A,k}(T^{-k}).\end{aligned}$$

□

Problem 6. Fix a number $n\geqslant 1$ and let $\mathrm{GL}_n(\mathbb{R}) = NAK$ and $\mathrm{SL}_n(\mathbb{R}) = NA_1K_1$ be the Iwasawa decompositions defined in class.

a) Let $B = (\mathbb{R}^{>0})^{n-1}$ (with Haar measure $\mathrm{d}^\times\mathfrak{b} = \prod_i\mathfrak{b}_i^{-1}\mathrm{d}\mathfrak{b}_i$) and consider the isomorphism $B\to A_1$ sending $\mathfrak{b}\in B$ to $\mathfrak{a}\in A_1$, where $\mathfrak{b}_i^n = \mathfrak{a}_{i+1}/\mathfrak{a}_i$ and conversely $\mathfrak{a}_i = (\mathfrak{b}_1\cdots\mathfrak{b}_{i-1})^n/(\mathfrak{b}_1^{n-1}\cdots\mathfrak{b}_{n-1})$. Show that the pull-back of the Haar measure on $\mathrm{SL}_n(\mathbb{R})$ defined in class along the diffeomorphism $N\times B\times K_1\to\mathrm{SL}_n(\mathbb{R})$ arising from the Iwasawa decomposition is as follows (with $(\mathfrak{n},\mathfrak{b},\mathfrak{k})\in N\times B\times K_1$):

$$n^{n-2}\cdot\prod_{j=1}^{n-1}\mathfrak{b}_j^{-nj(n-j)}\mathrm{d}^\times\mathfrak{n}\mathrm{d}^\times\mathfrak{b}\mathrm{d}^\times\mathfrak{k}.$$

Solution. Consider the diffeomorphisms

$$\mathbb{R}^{>0} \times N \times B \times K_1 \to \mathbb{R}^{>0} \times \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n^+(\mathbb{R}) \to N \times A \times K_1,$$

where the second map is given by $(\lambda, h) \mapsto \lambda h$. The pullback of $\prod_{i=1}^n \mathfrak{a}_i^{n+1-2i} \mathrm{d}^\times \mathfrak{n} \mathrm{d}^\times \mathfrak{a} \mathrm{d}^\times \mathfrak{k}$ along the third map is $\mathrm{d}^\times g$. The pullback of $\mathrm{d}^\times g$ along the second map is $n \mathrm{d}^\times \lambda \mathrm{d}^\times h$. It therefore suffices to prove that the pullback along the composition is

$$n^{n-1} \cdot \prod_{j=1}^{n-1} \mathfrak{b}_j^{-nj(n-j)} \mathrm{d}^\times \mathfrak{n} \mathrm{d}^\times \mathfrak{b} \mathrm{d}^\times \mathfrak{k}.$$

The composition $\mathbb{R}^{>0} \times N \times B \times K_1 \to N \times A \times K_1$ arises from the diffeomorphism $\mathbb{R}^{>0} \times B \to A$ given by $(\lambda, \mathfrak{b}) \mapsto \mathfrak{a}$ with $\mathfrak{a}_i = \lambda \cdot (\mathfrak{b}_1 \cdots \mathfrak{b}_{i-1})^n / (\mathfrak{b}_1^{n-1} \cdots \mathfrak{b}_{n-1})$. We therefore only need to show that the pullback of $\prod_{i=1}^n \mathfrak{a}_i^{n+1-2i} \mathrm{d}^\times \mathfrak{a}$ along this diffeomorphism is $n^{n-1} \cdot \prod_{j=1}^{n-1} \mathfrak{b}_j^{-nj(n-j)} \mathrm{d}^\times \lambda \mathrm{d}^\times \mathfrak{b}$. The inverse map is given by $\lambda = (\prod_i \mathfrak{a}_i)^{1/n}$, $\mathfrak{b}_i = (\mathfrak{a}_{i+1}/\mathfrak{a}_i)^{1/n}$. Note that $\prod_j \mathfrak{b}_j^{-nj(n-j)} = \prod_i \mathfrak{a}_i^{n+1-2i}$. We therefore want to show that the pullback of $\prod_i \mathrm{d}^\times \mathfrak{a}$ is $n^{n-1} \cdot \mathrm{d}^\times \lambda \mathrm{d}^\times \mathfrak{b}$. Applying log to $\lambda, \mathfrak{a}_i, \mathfrak{b}_i$, the map $A \to \mathbb{R}^{>0} \times B$ turns into $\log \mathfrak{a} \mapsto (\log \lambda, \log \mathfrak{a})$, where $\log \lambda = \frac{1}{n} \sum_i \log \mathfrak{a}_i$ and $\log b_i = \frac{1}{n}(\log \mathfrak{a}_{i+1} - \log \mathfrak{a}_i)$. This is a linear map described by the $n \times n$-matrix

$$\frac{1}{n} \cdot \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}$$

with determinant $n^{-(n-1)}$. The result follows since $\mathrm{d}^\times x = \mathrm{d} \log x$. $\square$

b) Let $\widetilde{\mathcal{F}} = N'A'K \subset \mathrm{GL}_n(\mathbb{R})$ be Siegel's almost fundamental domain for $\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R})$. Compute the volume of $\widetilde{\mathcal{F}} \cap \mathrm{SL}_n(\mathbb{R})$ with respect to the Haar measure on $\mathrm{SL}_n(\mathbb{R})$ defined in class.

Solution. Hopefully, by a), the definition of Siegel's almost fundamental domain and the fact that $\mathrm{vol}(\mathrm{SO}_n(\mathbb{R})) = \frac{1}{2} \cdot V_1 \cdots V_n$, we have

$$\mathrm{vol}(\widetilde{\mathcal{F}} \cap \mathrm{SL}_n(\mathbb{R})) = \frac{V_1 \cdots V_n}{2n(n-1)!^2} \cdot \left(\frac{2}{\sqrt{3}}\right)^{n(n-1)(n+1)/6},$$

where V_k is the volume the $(k-1)$-dimensional unit sphere S^{k-1}. Note that for $n \to \infty$, the volume rapidly goes to infinity. On the other hand, the volume $\zeta(2) \cdots \zeta(n)$ of a fundamental domain for $\mathrm{SL}_n(\mathbb{Z}) \backslash \mathrm{SL}_n(\mathbb{R})$ converges for $n \to \infty$. Hence, Siegel's fundamental domain is in this sense a very crude approximation to an actual fundamental domain! □

Problem 7. Don't worry, be happy.