Math 286X: Arithmetic Statistics

Spring 2020

Problem set #6

Problem 1. Let $n \geqslant 1$.

a) Show that, up to isomorphism, there are exactly $\lfloor \frac{n}{2} \rfloor + 1$ degree n extensions K of $\mathbb{R}$.

b) ([Bha07, Proposition 2.4]) Show that

$$\sum_{\substack{\text{degree } n \\ \text{extension } K|\mathbb{R}}} \frac{1}{\# \operatorname{Aut}_{\mathbb{R}}(K)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^k \cdot k!(n-2k)!} = \mathbb{P}(\pi^2 = \mathrm{id} \mid \pi \in S_n).$$

Problem 2. Let K be a nonarchimedean local field with prime ideal $\mathfrak{p}$ and residue field $\mathbb{F}_q$.

a) Show that $\int_{\mathcal{O}_K} |x| \mathrm{d}x = 1 - \frac{1}{q+1}$.

b) Let $f(X) \in \mathcal{O}_K[X]$ be a polynomial such that $f'(X) \bmod \mathfrak{p}$ has k simple roots in $\mathbb{F}_q$ and no roots of higher multiplicity in $\mathbb{F}_q$. For any $y \in \mathcal{O}_K$, let $m(y)$ be the number of $x \in \mathcal{O}_K$ such that $f(x) = y$. Show that

$$\int_{\mathcal{O}_K} m(y) \mathrm{d}y = 1 - \frac{k}{q+1}.$$

(This is the expected number of preimages of a random element $y \in \mathcal{O}_K$ under the map $f : \mathcal{O}_K \to \mathcal{O}_K$.)

Problem 3 ([Ser78, Section 4])**.** Let K be a local field with normalized valuation v_K and let $n \geqslant 1$.

a) Show that the discriminant of an Eisenstein polynomial $f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in \mathcal{O}_K[X]$ with $a_n = 1$ satisfies

$$v_K(\mathrm{disc}(f)) = \min_{1 \leqslant i \leqslant n} (i - 1 + n v_K(i a_i)).$$

b) Show that K has infinitely many separable totally ramified field extensions of degree n if and only if $\mathrm{char}(K) \mid n$.

c) Show that K has infinitely many field extensions of degree n if and only if $\mathrm{char}(K) \mid n$.

d) (bonus) Let $d \geqslant 0$. Show that K has a totally ramified field extension L of degree n with $v_K(D_{L|K}) = d$ if and only if

$$n \cdot v_K(l) \leqslant d - n + 1 \leqslant n \cdot v_K(n),$$

where $1 \leqslant l \leqslant n$ with $l \equiv d + 1 \mod n$.

e) (bonus) Compute the number of totally ramified field extensions $L \subset K^{\mathrm{sep}}$ of K of degree n with $v_K(D_{L|K}) = d$.

Problem 4. Let S_1 be a degree n_1 extension and let S_2 be a degree n_2 extension of a Dedekind domain R.

a) Show that the tensor product $S = S_1 \otimes_R S_2$ is a degree $n_1 \cdot n_2$ extension of R.

b) Show that $\mathrm{disc}(S|R) = \mathrm{disc}(S_1|R)^{n_2} \cdot \mathrm{disc}(S_2|R)^{n_1}$. (Hint: Look up the discriminant of a Kronecker product of matrices or the proof of Proposition I.2.11 in [Neu99]. First show the claim for principal ideal domains R.)

References

[Bha07] Manjul Bhargava. "Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants". In: *Int. Math. Res. Not. IMRN* 17 (2007), Art. ID rnm052, 20. ISSN: 1073-7928. DOI: 10.1093/imrn/rnm052. URL: https://doi.org/10.1093/imrn/rnm052.

[Neu99] Jürgen Neukirch. *Algebraic number theory.* Vol. 322. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. Springer-Verlag, Berlin, 1999, pp. xviii+571. ISBN: 3-540-65399-6. DOI: 10.1007/978-3-662-03983-0. URL: https://doi-org.ezp-prod1.hul.harvard.edu/10.1007/978-3-662-03983-0.

[Ser78] Jean-Pierre Serre. “Une “formule de masse” pour les extensions totalement ramifiées de degré donné d’un corps local”. In: *C. R. Acad. Sci. Paris Sér. A-B* 286.22 (1978), A1031–A1036. ISSN: 0151-0509.