Nonabelian group cohomology

<u>Def</u> Let G be a finite group. A <u>G-group</u> is a group A with a left action of G.

Define the group

$$H^0(G,A) = A^G = \{ a \in A \mid ga = a \ \forall g \in G \}.$$

Let $Z^1(G,A)$ be the set of <u>1-cocycles</u>:

maps $\varphi : G \to A$ s.t. $\varphi(gh) = \varphi(g) \cdot g\varphi(h)$ $\forall g, h \in G$.

Define an action of A on $Z^1(G,A)$ by

$$(a\varphi)(g) = a \cdot \varphi(g) \cdot ga^{-1} \text{ for } a \in A, \varphi \in Z^1(G,A), g \in G.$$

The set $B^1(G,A)$ of <u>1-coboundaries</u> is the A-orbit consisting of maps φ of the form $(g \mapsto a \cdot g^{-1}a)$ for some $a \in A$.

Define the <u>pointed set</u> $H^1(G,A) = A \backslash Z^1(G,A)$ with <u>base point</u> $1 = B^1(G,A)$.

[$H^2, H^3, \dots$ are problematic!]

<u>Rmk</u> If G acts trivially on A $(ga = a \ \forall g, a)$,

then $H^0(G, A) = A$,

$$Z^1(G, A) = \mathrm{Hom}_{\text{group}}(G, A)$$

and A acts on $Z^1(G, A)$ by conjugation:

$$(a\varphi)(g) = a\,\varphi(g)\,a^{-1}.$$

$$H^1(G, A) = {}_{A}\backslash^{\mathrm{Hom}(G, A)}.$$

<u>Rmk</u>

You get functoriality, "truncated long ex. seq.", etc.

cf. Milne: ~~a~~ Algebraic groups, Lie groups, and their arithmetic subgroups, chapter VI.

Nonabelian Galois cohomology

Def Let $L|K$ be a Galois ext. with Galois group G and A be a G-group such that

$$A = \bigcup_{\substack{F \subseteq L \\ \text{fin. subext. of } K}} A^{Gal(L|F)} .$$

Write $H^0(L|K, A) = H^0(G, A) = A^G$.

If $L|K$ is a fin. ext., let

$$H^1(L|K, A) = H^1(G, A).$$

For infinite extensions, let

$$H^1(L|K, A) = \varinjlim_{\substack{F \subseteq L \\ \text{fin. Gal. ext. of } K}} H^1(F|K, A^{Gal(L|F)}),$$

or define cocycles requiring that the map $\varphi: G \to A$ is continuous, where G comes with the Krull topology and A comes with the discrete topology.

Thm $H^1(L|K, L^\times) = 1$ (Hilbert 90)

$H^1(L|K, L) = 0$ (additive Hilbert 90)

$H^1(L|K, GL_n(L)) = 1$

(idea: $GL_n(L) = \mathrm{Aut}_L(L^n)$

$\rightsquigarrow$ el. of $H^1(L|K, GL_n(L))$ are in bij. with (n-dim.) K-vector spaces V (up to isom.) such that $V \otimes_K L \cong L^n$. But of course there is only one n-dimensional K-vector space!)

$$\boxed{\begin{array}{l} 1 \to SL_n(L) \to GL_n(L) \to L^\times \to 1 \\ 1 \to SL_n(K) \to GL_n(K) \to K^\times \\ \quad \to H^1(SL_n(L)) \to H^1(GL_n(L)) = 1 \end{array}}$$

$\hookrightarrow H^1(L|K, SL_n(L)) = 1$

$H^1(L|K, L[G]^\times) = 1$

(idea: $L[G]^\times = \mathrm{Aut}_{\text{left } L[G]\text{-mod.}}(L[G])$

$\rightsquigarrow$ el. of $H^1(L|K, L[G]^\times)$ are in bij. with left $K[G]$-mod. V such that $V \otimes_K L \cong L[G]$. But there is only one such V, namely $V = K[G]$, by an argument similar to our pf of the normal basis thm).

Thm (Generalized Kummer Theory, cf. Wright-Yukie: Prehomogeneous vector spaces and field extensions)

Let $L|K$ be a Galois ext.

Let $\mathcal{G}$ be an algebraic group defined over K

$$\big(\text{e.g. } \mathcal{G} = GL_n,\ SL_n,\ \underset{\substack{\uparrow\\ \mathbb{G}_m(K) = K^\times}}{\mathbb{G}_m},\ \ldots\big)$$

let $\mathcal{V}$ be a variety defined over K

$$\big(\text{e.g. } \mathcal{V} = \underset{\substack{\uparrow\\ \mathbb{A}^n(K) = K^n}}{\mathbb{A}^n}\big)$$

and consider an algebraic action of $\mathcal{G}$ on $\mathcal{V}$ defined over K. Assume that $H^1(L|K, \mathcal{G}(L)) = 1$.

Let $v_0 \in \mathcal{V}(K)$. Then, we obtain a bijection

$$H^1(L|K, \mathrm{Stab}_{\mathcal{G}(L)}(v_0)) \longleftrightarrow \mathcal{G}(K)\backslash\big(\mathcal{G}(L).v_0 \cap \mathcal{V}(K)\big)$$

$$\big(\underset{\substack{\uparrow\\ \mathrm{Gal}(L|K)}}{\sigma} \longmapsto g^{-1}\sigma(g)\big) \longleftrightarrow \underset{(g \in \mathcal{G}(L))}{\mathcal{G}(K)\, g.v}$$

Rmk If $F := \mathrm{Stab}_{\mathcal{G}(L)}(v_0)$ is contained in $\mathcal{G}(K)$,

then $H^1(L|K, F) = G \backslash \mathrm{Hom}_{cont}(\mathrm{Gal}(L|K) \longrightarrow F)$.

$\uparrow$ trivial action of $\mathrm{Gal}(L|K)$

If $L = K^{sep}$, then RHS $\leftrightarrow$ {nondeg. F-ext. of K}.

Pf of Thm

Every el. of $H^1(L|K, \mathrm{Stab})$ is of the form $(\sigma \mapsto g^{-1}\sigma(g))$ with $g \in \mathcal{G}(L)$ because the image in $H^1(L|K, \mathcal{G}(L)) = 1$ is trivial, so a 1-coboundary in $B^1(L|K, \mathcal{G}(L))$.

We have

$$\mathcal{G}(K)\, g \cdot v_0 = \mathcal{G}(K)\, g' \cdot v_0$$

$$\Updownarrow$$

$$\mathcal{G}(K)\, g\, \mathrm{Stab} = \mathcal{G}(K)\, g'\, \mathrm{Stab}$$

$$\Updownarrow$$

$$\exists\, h \in \mathcal{G}(K),\ s \in \mathrm{Stab} : g' = h g s$$

$$(\sigma \mapsto g^{-1} \sigma(g)) = (\sigma \mapsto g'^{-1} \sigma(g')) \text{ in } H^1(L|K, \mathrm{Stab})$$

$$\Updownarrow$$

$$\exists\, s \in \mathrm{Stab} : \forall \sigma : s^{-1} g^{-1} \sigma(g) \sigma(s) = g'^{-1} \sigma(g')$$

$$\Updownarrow$$

$$\exists\, s \in \mathrm{Stab} : \forall \sigma : g' s^{-1} g^{-1} = \sigma(g' s^{-1} g^{-1})$$

$$\Updownarrow$$

$$\exists\, s \in \mathrm{Stab} : g' s^{-1} g^{-1} \in \mathcal{G}(K).$$

...

$\square$

Example (Kummer Theory)

K field of $\mathrm{char}(K) \nmid n$ and $\zeta_n \in K$.

$\mathcal{G} = \mathbb{G}_m \rightsquigarrow \mathcal{G}(L) = L^\times \rightsquigarrow H^1(\Gamma_K, \mathcal{G}(L)) = 1$ by H 90

$\mathcal{V} = \mathbb{G}_m$

$\mathcal{G} \curvearrowright \mathcal{V}: \quad x \cdot y = x^n y$

$v_0 = 1 \in K^\times = \mathcal{V}(K)$

$$\mathrm{Stab}_{\mathcal{G}(K^{sep})}(v_0) = \langle \zeta_n \rangle \underset{\substack{\uparrow \\ \zeta_n \in K}}{\subseteq} \mathcal{G}(K)$$

$\langle \zeta_n \rangle \cong C_n$ ($\mathrm{char}(K) \nmid n$)

$$\mathcal{G}(K^{sep}) \cdot v_0 = (K^{sep})^{\times n} \underset{\substack{\uparrow \\ \mathrm{char}(K) \nmid n}}{=} (K^{sep})^\times .$$

Hence,

$$\{C_n\text{-ext. of } K\} \longleftrightarrow \mathcal{G}(K) \backslash \left((K^{sep})^\times \cap K^\times \right) .$$

$$= \mathcal{G}(K) \backslash K^\times$$

$$= K^{\times n} \backslash K^\times$$

Example (cubic ext.)

K any field of $\mathrm{char}(K) \neq 6$.

$\mathcal{G} = GL_2$

$V = \{\text{binary cubic forms } f(x,y)\}$

action $\mathcal{G} \circlearrowright V$ as before:

$$(M.f)(v) = \frac{f(M^T v)}{\det(M)}$$

$v_0 = XY(X-Y)$

$$\mathrm{Stab}_{GL_2(K^{sep})}(v_0) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \right\rangle \subseteq GL_2(K)$$
$$\cong S_3$$

$GL_2(K^{sep}).v_0 = V^{disc \neq 0}(K^{sep})$

$= \{f \in V(K^{sep}) \mid \mathrm{disc}(f) \neq 0\}$ is

a dense subset of $V(K^{sep})$. (Note that $\dim(GL_2) = 4 = \dim(V)$.) Hence,

$$\{S_3\text{-ext. of } K\} \leftrightarrow GL_2(K) \backslash V^{disc \neq 0}(K).$$

Example (deg. 4 ext.)

k any field of char $(k) \nmid 4!$.

$\mathcal{G}' = GL_2 \times GL_3$

$$V(L) = L^2 \otimes \underbrace{\mathrm{Sym}^2(L^3)}_{\{\text{symm. } 3\times 3\text{-matrices}\}}$$

$\mathcal{G}' \curvearrowright V$: $(M, N).(a \otimes B) = (Ma) \otimes (N B N^T)$.

This action has kernel

$$T = \{(\lambda^2 I_2, \lambda^{-1} I_3) \mid \lambda \in \mathbb{G}_m\}.$$

Let $\mathcal{G} = \mathcal{G}'/T$. $\rightsquigarrow$ $\mathcal{G} \curvearrowright V$

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} \mathrm{Stab}_{\mathcal{G}(k^{sep})}(v_0) & \subseteq & \mathcal{G}(k) \\ \text{\rotatebox{90}{$\cong$}} & & \\ S_4 & & \end{array}$$

$\mathcal{G}(k^{sep}).v_0$ is a dense subset of $V(k^{sep})$

(Note that $\dim(\mathcal{G}) = 4 + 9 - 1 = 12 = 2 \cdot 6 = \dim(V)$.)

$$\{S_4\text{-ext. of } k\} \longleftrightarrow \mathcal{G}(k) \backslash \underbrace{(\mathcal{G}(k^{sep}).v_0 \cap V(k))}_{\approx V(k)}.$$

Example (deg. 5 ext.)

$$\mathcal{G}' = GL_4 \times GL_5$$

$$V(L) = L^4 \otimes \underbrace{\mathrm{Alt}^2(L^5)}_{\{\text{skew-symm. } 5 \times 5\text{-matrices}\}}$$

$$\mathcal{G}' \curvearrowright V : \quad (M, N).(a \otimes B) = (Ma) \otimes (N B N^T).$$

$\mathcal{G} := \mathcal{G}'/T$ where $T = \{(\lambda^2 I_4, \lambda^{-1} I_5) \mid \lambda \in \mathbb{G}_m\}$.

$$\mathcal{G}(K) \supseteq \mathrm{Stab} \cong S_5$$

$$\dim(\mathcal{G}) = 16 + 25 - 1 = 40 = 4 \cdot 10 = \dim(V)$$

$$\boxed{\text{THE END}}$$