MCS is a statement about the number of field extensions:

Rmk Let $d : G \to \mathbb{R}^{\geq 0} \cup \{\infty\}$. Denote the restriction to $H \subseteq G$ by $d|_H$.

Always $a(d|_H) \geq a(d)$.

But sometimes $a(d|_H) = a(d)$ and $b(d|_H, K) > b(d, K)$.

$\Rightarrow$ By MCS,

grows more slowly

$$\#\{\text{field } G\text{-ext. } L|K : \text{inv}(L) \leq T\} \sim C_1 \cdot T^{\hat{a}} (\log T)^{b(d)-1}$$

$$\#\{\text{field } H\text{-ext. } L'|K : \text{inv}(L') \leq T\} \sim C_2 \cdot T^{\hat{a}} (\log T)^{b(d|_H)-1}$$

$\wedge \leftarrow L := \text{Ind}_H^G L'$

$$\#\{\text{nondeg. } G\text{-ext. } L|K : \text{inv}(L) \leq T\}$$

grows more quickly

Ex There are $\sim C_1 \cdot T$ deg. 4 field ext. $L|\mathbb{Q}$ with $\text{disc}(L) \leq T$. $(G = S_4)$

There are $\sim C_2 \cdot T(\log T)$ products L of two degree 2 field ext. of $\mathbb{Q}$ with $\text{disc}(L) \leq T$. $(H = S_2 \times S_2 \subseteq S_4)$.

Heuristic for MCS

(cf. Malle: On the distribution of Galois groups,
" , II)

Assume there exists a field G-ext. $L|K$
with $\mathrm{inv}(L) < \infty$.

Basic assumption: For a finite set of places S
and nondeg. G-ext. L_v of K_v $(v \in S)$, the
number of field G-ext. L of K that are
unramified at all $v \notin S$ and such that
$L \otimes_K K_v \cong L_v$ for all $v \in S$
is "on average" $C_1 \cdot \prod_v \frac{1}{\# \mathrm{Aut}_{G\text{-ext}}(L_v)}$

for some constant $C_1 > 0$.

["local-global principle"]

In terms of Dirichlet series:

let $$D(s) = \sum_{\substack{L|K \text{ field} \\ G\text{-ext.}}} \frac{\mathrm{inv}(L)^{-s}}{\# \mathrm{Aut}(L)}$$

and $$D_v'(s) = \sum_{\substack{L_v|K_v \text{ nondeg.} \\ G\text{-ext.}}} \frac{\mathrm{inv}_v(L_v)^{-s}}{\# \mathrm{Aut}(L_v)}.$$

⇝ Basic assumption (basically by Tauberian thm / Wiener–Ikehara)

$$D(s) \approx \prod_v D'_v(s)$$

both sides have rightmost pole at the same positions and of the same order

$$\Rightarrow D(s) \approx \prod_v D'_v(s)$$

$$\approx \prod_{\substack{\mathfrak{q} = v \\ \text{not exceptional}}} D'_v(s)$$

not exceptional [i.e. $\mathfrak{q} \nmid |G|$, inv$_v$ given by d]

each $D'_v(s)$ is a finite sum and therefore entire

$$\approx \prod_{\mathfrak{q} \nmid |G|} \sum_{\substack{\text{conj. cl. } C: \\ C = C^{\mathrm{Nm}(\mathfrak{q})}}} (\mathrm{Nm}(\mathfrak{q})^{d(C)})^{-s}$$

cf. counting tamely ram. ext. of local fields

$$\approx \prod_{\mathfrak{q} \nmid |G|} \Big(\underset{\substack{\uparrow \\ C = \{id\} \\ (\text{unram. ext. of } K_v)}}{1} + \sum_{\substack{C: \\ C = C^{Nm(\mathfrak{q})}, \\ d(C) = a}} Nm(\mathfrak{q})^{-as} \Big)$$

$$\approx \prod_{\mathfrak{q} \nmid |G|} \prod_{\substack{C: \\ C = C^{Nm(\mathfrak{q})} \\ d(C) = a}} \left(1 + Nm(\mathfrak{q})^{-as}\right).$$

The Frobenius automorphism in
$U = Gal(K(\zeta_{|G|})|K) \subseteq (\mathbb{Z}/|G|\mathbb{Z})^\times$
of $\mathfrak{q} \nmid |G|$ with residue field $\mathbb{F}_q$ $(q = Nm(\mathfrak{q}))$
is $(q \bmod |G|)$. Hence,

$C = C^{Nm(\mathfrak{q})} \iff C = C^q \iff (q \bmod |G|) \in Stab_U(C)$

$\iff Frob(\mathfrak{q}) \in Stab_U(C)$

$\iff Frob(\mathfrak{q})$ maps to id in $U/Stab_U(C)$
$$\cong Gal(K(\zeta)_C | K),$$

where $K(\zeta)_C := K(\zeta)^{Stab_U(C)}$

$(\Leftrightarrow)$ y splits completely in $K(\mathfrak{J})_c$.

Therefore,

$$D(s) \approx \prod_{\substack{C: \\ d(C)=a}} \prod_{\substack{y \nmid |G| \\ \text{splitting} \\ \text{completely} \\ \text{in } K(\mathfrak{J})_c}} \left(1 + \mathrm{Nm}(y)^{-as}\right)$$

$$= \prod_{\substack{U\text{-orbit }[C]: \\ d(C)=a}} \prod_{\substack{y \nmid |G| \\ \text{s.c. in} \\ K(\mathfrak{J})_c}} \left(1 + \mathrm{Nm}(y)^{-as}\right)^{[U : \mathrm{Stab}_U(C)]}$$

U abelian,
so every el. of
an orbit has
the same stabilizer

$$= \qquad \qquad - \left(\qquad \right)^{[K(\mathfrak{J})_c : K]}$$

$$= \underbrace{\prod_{\substack{U\text{-orbit }[C]: \\ d(C)=a}} \underbrace{\zeta_{K(\mathfrak{J})_c}(as)}_{\text{rightmost: simple pole at } s = \frac{1}{a}}}_{\text{order } b (= \# U\text{-orbits}) \text{ pole at } s = \frac{1}{a}}.$$

Dedekind
zeta function

$\Rightarrow$ By Tauberian theorems / Wiener-Ikehara,

$$\sum_{\substack{L|K \text{ field} \\ G\text{-ext.} \\ \operatorname{inv}(L) \le T}} \underbrace{\frac{1}{\#\operatorname{Aut}(L)}}_{= \frac{1}{\#(\text{center of } G)}} \sim C \cdot T^{\frac{1}{a}} (\log T)^{b-1}.$$

"$\square$"

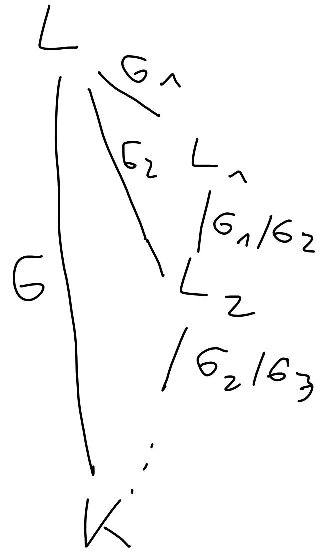
Strategies for proving (special cases of) MCS

1. Understand quotients of Γ_K.

   E.g. classfield theory description of the abelianization Γ_K^{ab}, dealing with abelian groups G

   (Wright: Distribution of discriminants of abelian extensions,
   Wood: On the probabilities of local behaviors in abelian field extensions)

2. Induction along chains of normal subgroups $1 \leq G_1 \leq G_2 \leq \dots \leq G_n = G$.

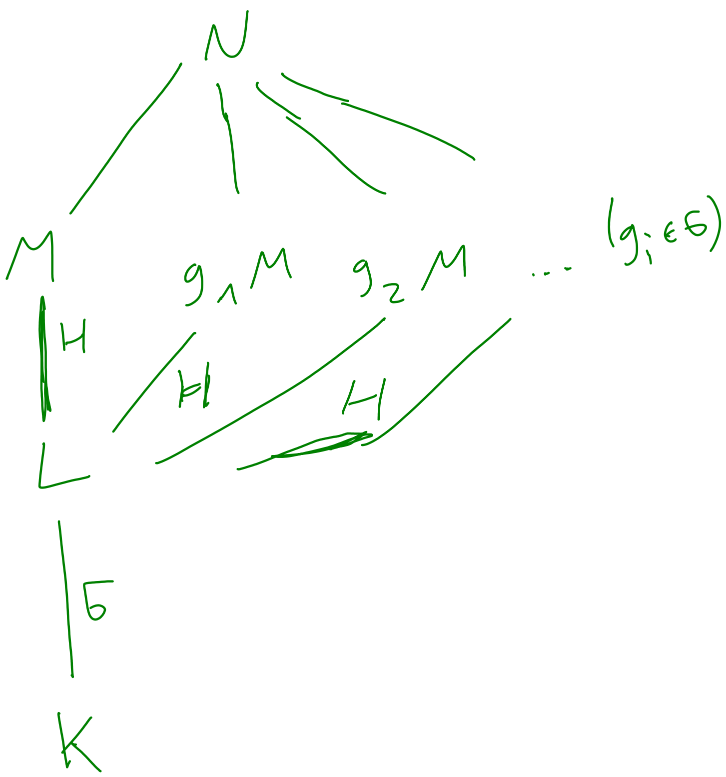


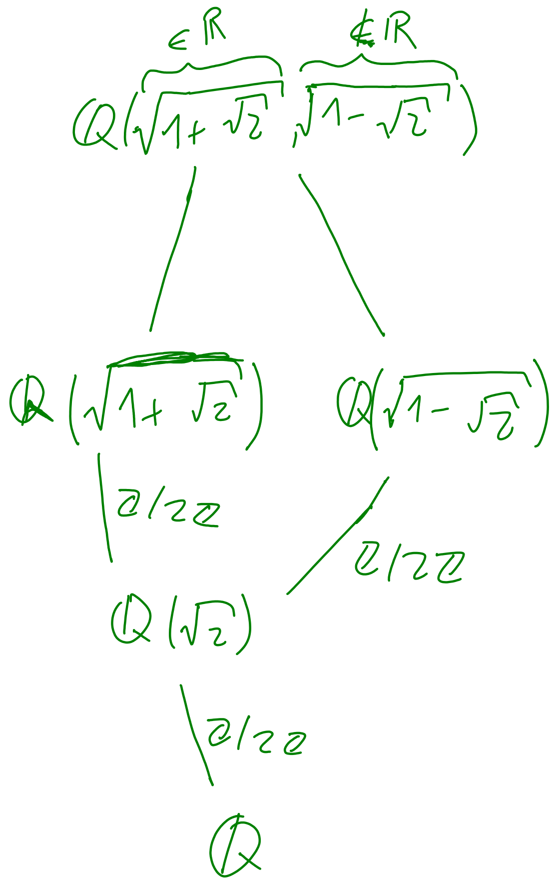
Try solving the problem for all G_{i+1}/G_i, then using induction.

Klüners proved MCS for nilpotent G when inv = disc in his habilitation: "Über die Asymptotik von Zahlkörpern mit vorgegebener Galoisgruppe".

Side remark:

Let $L|K$ be a Gal. ext. with Galois group G and let $M|L$ be a Gal. ext. with Galois group H. Let N be the Galois closure of $M|K$. Then, $\mathrm{Gal}(N|K)$ is a subgroup of the <u>wreath product</u> $H \wr G = \left(\prod_{g\in G} H\right) \underset{\text{permutation action}}{\rtimes} G$.





It is in fact a <u>transitive</u> subgroup:

The composition

$\mathrm{Gal}(N|K) \hookrightarrow H \wr G \twoheadrightarrow G$ is surjective.

(This is analogous to the fact that the Galois group of the Galois closure of a degree n field extension is a transitive subgroup of S_n).

3. Analyze the mult. table of a G-ext. w.r.t. a normal basis (similar to what we did for cubic ext.)

(cf. Bhargava: The density of discriminants of quartic/quintic rings and fields)

4. Generalized Kummer theory

(cf. Wright-Yukie: Prehomogeneous vector spaces and field extensions)