G-extensions

Let G be a finite group.

Def A <u>G-ext.</u> L of a field K is a degree $|G|$ ext. of K with a left action of G on the K-algebra L (fixes $1 \in L$, satisfies $g(x+y) = gx + gy$, $g(x \cdot y) = (gx)\cdot(gy)$, $g(\lambda x) = \lambda g x$ for $\lambda \in K$), which has a <u>normal basis</u>: a K-basis of the form

$$(g\alpha)_{g \in G} \text{ for some } \alpha \in L.$$

An <u>isom. of G-ext.</u> is a G-equivariant isom. $L_1 \xrightarrow{\sim} L_2$ of K-algebras.

Rmk (base change)
If L is a G-ext. of K and $K'|K$ is any field ext., then $L \otimes_K K'$ is a G-ext. of K'.

Rmk We can regard L a left $K[G]$-module.

Then, $\exists$ normal basis $(g\alpha)_{g\in G}$ $\iff$ $L \cong K[G]$ as a left $K[G]$-module, $g\alpha \leftrightarrow g$

Ex The <u>trivial G-ext.</u> $L = \prod_{g \in G} K = K \times \dots \times K$

with G-action $g(x_{g'})_{g' \in G} = (x_{g^{-1}g'})_{g' \in G}$.

Equivalently: $L = K[G]$ with mult. in L given by

$$\Big(\sum_g x_g g\Big) \cdot_L \Big(\sum_g y_g g\Big) = \sum_g x_g y_g g.$$

Thm (Normal basis theorem)

A Galois ext. $L|K$ with Galois group G has a normal basis (i.e. is a G-ext.)

Pf (when $|K| = \infty$)

Let $G = \{g_1, \dots, g_n\}$.

$L \otimes K^{sep}$ is a nondeg. deg. $|G|$ ext. of K^{sep}.

$$\Rightarrow L \otimes K^{sep} \cong \underbrace{K^{sep} \times \dots \times K^{sep}}_{|G|}$$

$$L \hookrightarrow L \otimes K^{sep}$$

$\rightsquigarrow$ We obtain n distinct embeddings $L \hookrightarrow K^{sep}$, corr. to the n aut. of $L \subseteq K^{sep}$.

$$\Rightarrow \quad L \hookrightarrow K^{sep} \times \dots \times K^{sep}$$
$$y \mapsto (g_1 y, \dots, g_n y)$$

$\Rightarrow$ $L \otimes K^{sep}$ is the triv. G-ext of K^{sep}

Now, fix a K-basis $\omega_1, \dots, \omega_n$ of L.

Consider the pol.

$$f(X_1,\dots,X_n) = \det\left(g_i g_j (X_1\omega_1+\dots+X_n\omega_n)\right)_{i,j}$$
$$= \det\left(X_1 g_i g_j \omega_1 + \dots + X_n g_i g_j \omega_n\right)_{i,j}.$$

For $a_1,\dots,a_n \in K$, $\alpha = a_1\omega_1+\dots+a_n\omega_n \in L$, the following are equivalent:

$(g_i\alpha)_i$ is a basis of L

$\Leftrightarrow$ The image $((g_j g_i \alpha)_j)_i$ is a basis of $K^{sep}\times\dots\times K^{sep}$.

$\Leftrightarrow \det(g_i g_j \alpha)_{i,j} \neq 0$

$\Leftrightarrow f(a_1,\dots,a_n) \neq 0$.

Hence,

L has a normal basis

$\Leftrightarrow f(a_1,\dots,a_n)\neq 0$ for some $a_1,\dots,a_n\in K$

$\Leftrightarrow$ pol. $f(X_1,\dots,X_n)\neq 0$

$|K|=\infty$

$\Leftrightarrow f(a_1,\dots,a_n)\neq 0$ for some $a_1,\dots,a_n \in K^{sep}$

$\Leftrightarrow L\otimes_K K^{sep}$ has a normal basis (true!)

same argument as before

$\square$

Def Let L be an H-ext. of K for some $H \leq G$.

Define the <u>induced G-ext</u> $\mathrm{Ind}_H^G L = K[G] \otimes_{K[H]} L$

with G-action $g(a \otimes b) = (ga) \otimes b$ and with

mult. given by

$$(g \otimes b) \cdot (g \otimes b') = g \otimes (bb')$$

$$(g \otimes b) \cdot (g' \otimes b') = 0 \quad \text{when } gH \neq g'H.$$

$$\Big[(g \otimes b) \cdot \underbrace{(gh \otimes b)}_{(g \otimes hb)} = g \otimes (b \cdot hb') \Big]$$

Rmk $\mathrm{Ind}_H^G L \cong \underbrace{L \times \ldots \times L}_{r := [G:H]}$ as a K-algebra.

Pf Choose repr. $g_1, \ldots, g_r$ of the cosets in G/H.

$g_1 \otimes b_1 + \ldots + g_r \otimes b_r \longmapsto (b_1, \ldots, b_r)$. $\square$

Ex $\mathrm{Ind}_1^G K \cong K \times \ldots \times K$ is the triv ext.

Ex $\mathrm{Ind}_G^G L = L$

Thm $\mathrm{Ind}_H^G L$ is a G-ext.

Pf $\mathrm{Ind}_H^G L = K(G) \otimes_{K(H)} L \underset{\substack{\uparrow \\ L \text{ is } H\text{-ext}}}{\cong} K(G) \otimes_{K(H)} K(H) \cong K[G]$

as a left $K[G]$-module. + check that you get a left action of G on the K-algebra! $\square$

Thm (Classification)

The nondeg. G-set. of K can be written as $L = \mathrm{Ind}_H^G F$, where $H \subseteq G$ and where F is a Galois ext. of K with Galois group H. The automorphism group is

$$\mathrm{Aut}_{G\text{-est.}}(L) \cong C_G(H) = \{ g \in G \mid \forall h \in H : gh = hg \},$$

the centralizer of H in G.

(Pf. skipped!)

Another way to look at G-extensions:

Def Let $\Gamma_K := \mathrm{Gal}(K^{sep} | K)$ be the absolute Gal. group of K. To any continuous surjective hom. $f : \Gamma_K \twoheadrightarrow G$, we associate the Galois ext.

$$L_f = (K^{sep})^{\ker(f)}$$ of K with Galois group G.

More generally, to any cont. hom. $f : \Gamma_K \to G$, we associate the G-ext. $L = \mathrm{Ind}^G_H \underbrace{(K^{sep})^{\ker(f)}}_{\substack{\text{Gal. ext. with} \\ \text{Gal. group } H}}$

where $H = \mathrm{im}(f)$.

Thm Let G act on $\mathrm{Hom}_{cont}(\Gamma_K \to G)$ by conjugation. We get a bij.

$$G \backslash \mathrm{Hom}_{cont}(\Gamma_K \to G) \longleftrightarrow \{\text{nondeg. } G\text{-ext. of } K \text{ (up to isom.)}\}$$

$$f \longmapsto L_f$$

Furthermore $\mathrm{Stab}_G(f) = C_G(\mathrm{im}(f)) \cong \mathrm{Aut}_{G\text{-ext}}(L_f)$.

Also, f is surjective if and only L_f is a field (= Gal. ext. of K).

Rmk If $K'|K$ is a separable field ext. and L is a G-ext. of K corr. to $f : \Gamma_K \to G$, then the G-ext. $L \otimes_K K'$ of K' corr. to $f' : \Gamma_{K'} \subseteq \Gamma_K \to G$.

Rmk Let G_1, G_2 be finite groups. We obtain a bij.

$$\{\text{nondeg. } G_1\text{-ext. of } K\} \times \{\text{nondeg. } G_2\text{-ext. of } K\} \leftrightarrow \{\text{nondeg. } G_1 \times G_2\text{-ext. of } K\}$$

$$(L_1, L_2) \longmapsto L_1 \otimes_K L_2$$

If L_i corr. to $f_i : \Gamma_K \to G_i$, then

$$L_1 \otimes L_2 \text{ corr. to } \begin{aligned} \Gamma_K &\longrightarrow G_1 \times G_2 \\ \sigma &\longmapsto (f_1(\sigma), f_2(\sigma)) \end{aligned} .$$

$$\mathrm{Aut}_{G_1\text{-ext}}(L_1) \times \mathrm{Aut}_{G_2\text{-ext}}(L_2) \cong \mathrm{Aut}(L_1 \otimes L_2)$$

If $L_1, L_2 \subseteq K^{sep}$ are fields, then

$$L_1 \otimes L_2 \cong \mathrm{Ind}_H^{G_1 \times G} (\underbrace{L_1 \cdot L_2}_{\text{compositum}}),$$

where $H = \mathrm{Gal}(L_1 \cdot L_2 | K) \subseteq G_1 \times G_2$.

<u>Rmk</u> Let $n \geq 1$. We obtain a bij.

$$\{\text{nondeg. deg. } n \text{ ext. of } K\} \longleftrightarrow \{\text{nondeg. } S_n\text{-ext. of } K\}$$
$$F = L^T \longleftarrow\!\!| \quad L$$

where $T < S_n$ is the set of perm. of $\{1, \dots, n\}$ that fix 1 and $L^T = \{x \in L \mid \forall g \in T : gx = x\}$.

[For the '$\to$' map, see Bhargava, Satriano: On a notion of "Galois closure" for extensions of rings.]

Let $\sigma_1, \dots, \sigma_n$ be the K-algebra homomorphisms $F \longrightarrow K^{sep}$.

Then, the map $f : \Gamma_K \longrightarrow S_n$ corr. to the S_n-ext. L represents the action of Γ_K on the set $\{\sigma_1, \dots, \sigma_n\}$ of n hom. (by composition).

$$\mathrm{Aut}_{K\text{-alg.}}(F) \cong \mathrm{Aut}_{S_n\text{-ext}}(L)$$

F is a field if and only if the action of Γ_K on $\{1, \dots, n\}$ (induced by $f : \Gamma_K \to S_n$) is transitive. Then, $(K^{sep})^{\ker(f)}$ is Galois closure of F/K.