Thm Let S be a nondeg. deg. n eset. of a Dedekind dom. R. Then:

S maximal $\Longleftrightarrow$ S maximal at every $\mathfrak{p}$

Pf "$\Leftarrow$" $S = \{x \in \underbrace{S \otimes K}_{L} \mid x \in S \otimes R_{\mathfrak{p}} \ \forall \mathfrak{p}\}$

$$= " \bigcap_{\mathfrak{p}} (S \otimes R_{\mathfrak{p}}) "$$

+ Rmk

"$\Rightarrow$" R is dense in $R_{\mathfrak{p}}$

$\Rightarrow$ $S = S \otimes_R R$ is dense in $S \otimes R_{\mathfrak{p}}$

+ Rmk $\square$

Rmk

max. $\Longleftrightarrow$ max. at every $\mathfrak{p}$

$\Uparrow$ $\Uparrow$

disc. sqfree $\Longleftrightarrow$ disc. sqfree at every $\mathfrak{p}$

($\mathfrak{p}^2 \nmid$ disc)

(Reiner, Maximal orders)

For cubic ext., denote by $V^{max}(R)$ the set of $f \in V_{disc \neq 0}(R)$ corr. to max. ext. S of R.

Big goal

<u>Thm</u> $N(T) := \sum\limits_{\substack{\text{deg. 3 field} \\ \text{ext. } L|\mathbb{Q} \\ \text{with } |D_L| \leq T}} \underbrace{\frac{1}{\# \mathrm{Aut}(L)}}_{\substack{=1 \text{ for} \\ 100\% \text{ of } L}} \sim \frac{1}{3\zeta(3)} \cdot T$ for $T \to \infty$.

In part. $\#\{\text{deg. 3 field ext. } L|\mathbb{Q} \text{ with } |D_L| \leq T\} \sim \frac{1}{3\zeta(3)} \cdot T$ for $T \to \infty$.

<u>Pf</u> Overview:

$$N(T) = \sum_{\substack{[f] \in GL_2(\mathbb{Z}) \backslash V^{irred, max}(\mathbb{Z}) \\ |disc(f)| \leq T}} \frac{1}{\# \mathrm{Stab}_{GL_2(\mathbb{Z})}(f)}$$

Let $\mathcal{F}_T$ be a fund. dom. for $GL_2(\mathbb{Z})\backslash V_{0\neq|\mathrm{disc}|\leq T}(\mathbb{R})$.

$\Rightarrow N(T) = \#(\mathcal{F}_T \cap V^{\mathrm{irred,\,max}}(\mathbb{Z}))$

Basically, this is a lattice-point counting problem. To reduce $V(\mathbb{Z})$ to $V^{\mathrm{irred,\,max}}(\mathbb{Z})$, use a sieve.

<u>Step 1: Construct a nice fund. dom. $\mathcal{F}_T$ for $GL_2(\mathbb{Z})\backslash V_{0\neq|\mathrm{disc}|\leq T}(\mathbb{R})$</u>

Recall the bij.

$$GL_2(\mathbb{R})\backslash V_{0\neq \mathrm{disc}}(\mathbb{R}) \longleftrightarrow \{\text{nondeg. cubic ext. of } \mathbb{R}\}$$

$$\{\text{nondeg. cubic ext. of } \mathbb{R}\} = \{\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \mathbb{R}\times\mathbb{C}\}$$

Let $f_1, f_2 \in V(\mathbb{R})$ correspond to $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $\mathbb{R}\times\mathbb{C}$,

$\Rightarrow \#\,\mathrm{Stab}_{GL_2(\mathbb{R})}(f_1) = \#\underbrace{\mathrm{Aut}(\mathbb{R}\times\mathbb{R}\times\mathbb{R})}_{S_3} = 6$

$\#\,\mathrm{Stab}_{GL_2(\mathbb{R})}(f_2) = \#\,\mathrm{Aut}(\mathbb{R}\times\mathbb{C}) = 2$

W.l.o.g. $|\mathrm{disc}(f_1)| = |\mathrm{disc}(f_2)| = 1$.

(E.g. $f_1 = XY(X+Y)$, $f_2 = \frac{1}{\sqrt{2}} X(X^2+Y^2)$.)

Now, $\mathcal{F}^{\mathbb{R}} := \{f_1\}^{\sqcup \frac{1}{6}} \sqcup \{f_2\}^{\sqcup \frac{1}{2}}$ is a

fund. dom. for $GL_2(\mathbb{R}) \backslash V_{\mathrm{disc} \neq 0}(\mathbb{R})$. (I)

Let $\mathcal{F}^{SL}$ be a fund. dom. for $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$.

$\Rightarrow \mathcal{F}^{GL^{\pm 1}} := \left(\mathcal{F}^{SL} \sqcup \mathcal{F}^{SL} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)^{\sqcup \frac{1}{2}}$

is a fund. dom for $GL_2(\mathbb{Z}) \backslash \underbrace{GL_2^{\pm 1}(\mathbb{R})}_{\{M \in GL_2(\mathbb{R}) :\ \det(M) = \pm 1\}}$.

$\Rightarrow \mathcal{F}^{GL}_T := (0, T^{1/4}] \cdot \mathcal{F}^{GL^{\pm 1}}$ is a fund.

dom. for $GL_2(\mathbb{Z}) \backslash \underbrace{GL_2^{|\det| \leq T^{1/2}}(\mathbb{R})}_{\{M \in GL_2(\mathbb{R}) \mid |\det(M)| \leq T^{1/2}\}}$.

(II)

$$(\mathrm{I}),(\mathrm{II}) \Rightarrow \mathcal{F}_T := \mathcal{F}_T^{GL} \cdot \mathcal{F}^{\mathbb{R}}$$

$$:= \bigsqcup_{M \in \mathcal{F}_T^{GL}} M . \mathcal{F}^{\mathbb{R}}$$

$$= \bigsqcup_{M \in \mathcal{F}_T^{GL}} \{M f_1\}^{\sqcup \frac{1}{6}} \sqcup \{M f_2\}^{\sqcup \frac{1}{2}}$$

is a fund. dom. for $GL_2(\mathbb{Z}) \backslash V_{0 \neq |\mathrm{disc}| \leq T}^{(\mathbb{R})}$

(because $|\mathrm{disc}(Mf)| = |\det(M)|^2 \cdot |\mathrm{disc}(f)|$

and $|\mathrm{disc}(f_1)| = |\mathrm{disc}(f_2)| = 1$).

Note: weight of f in $\mathcal{F}_T$:

$$\chi_{\mathcal{F}_T}(f) = \frac{1}{6} \# \{M \in \mathcal{F}_T^{GL} \mid M f_1 = f\}$$

(at least one of these is 0)

$$+ \frac{1}{2} \# \{M \in \mathcal{F}_T^{GL} \mid M f_2 = f\}.$$

Step 2: $\mathrm{vol}(\mathcal{F}_T) = \frac{1}{3}\zeta(2)\cdot T$

We've shown that the maps

$$\eta_{f_1}, \eta_{f_2} : GL_2(\mathbb{R}) \xrightarrow{\subset \mathbb{R}^4} V(\mathbb{R}) = \mathbb{R}^4$$
$$M \longmapsto Mf_1,\ Mf_2$$

have abs. Jac. det. $|\mathrm{disc}(f_1)|, |\mathrm{disc}(f_2)| = 1$ at M.

$$\Rightarrow \mathrm{vol}(\mathcal{F}_T) = \frac{1}{6}\mathrm{vol}(\eta_{f_1}(\mathcal{F}^{GL}_T) \text{ as a multiset})$$
$$+ \frac{1}{2}\mathrm{vol}(\eta_{f_2}(\mathcal{F}^{GL}_T) \text{ as a multiset})$$

$$= \left(\frac{1}{6}+\frac{1}{2}\right)\cdot \int_{\mathcal{F}^{GL}_T} 1\, d^+M$$

change of variables

$$= \frac{2}{3}\cdot \int_{\underbrace{\mathcal{F}^{GL}_T}_{(0,T^{1/4}]\cdot \mathcal{F}^{GL^{\pm 1}} = (0,T^{1/4}]\cdot\left(\mathcal{F}^{SL} \cup \mathcal{F}^{SL}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)^{\cup\frac{1}{2}}}} |\det(M)|^2\, d^\times M$$

$$d^\times M = \frac{d^+M}{|\det(M)|^2}$$

$$= \frac{2}{3} \cdot \int_0^{T^{1/4}} \int_{\mathcal{F}^{SL}} \underbrace{|\det(\lambda h)|^2}_{\lambda^4} \cdot 2 d^\times h \, d^\times \lambda$$

$M = \lambda h$
$\rightsquigarrow d^\times M = 2 d^\times \lambda \, d^\times h$
was the def. of our Haar measure $d^\times h$ on $SL_2(\mathbb{R})$

$$= \frac{2}{3} \cdot 2 \cdot \int_0^{T^{1/4}} \lambda^4 d^\times \lambda \cdot \int_{\mathcal{F}^{SL}} 1 \, d^\times h$$

$$= \frac{2}{3} \cdot 2 \cdot \frac{(T^{1/4})^4}{4} \cdot \mathrm{vol}(\mathcal{F}^{SL})$$

fund. dom. for $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$

$$= \frac{1}{3} \cdot T \cdot \zeta(2).$$

Step 3: Cut off cusp

Note: Take $n a k \in \mathcal{F}_{\text{Siegel}}$

$$\underbrace{n}_{\subset N' \text{ cpt.}} \; \underbrace{a}_{\subset A'} \; \underbrace{k}_{\subset K \text{ cpt.}}$$

$$A' = \left\{ \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \,\middle|\, t \geq \underbrace{\sqrt{\tfrac{\sqrt{3}}{2}}}_{?} \right\}$$

Fix $f = aX^3 + bX^2Y + cXY^2 + dY^3$.

$$\Rightarrow \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} f = \underbrace{t^{-3}}_{\to 0 \text{ for } t \to \infty} aX^3 + t^{-1} bX^2Y + tcXY^2 + t^3 dY^3$$

Let $f = aX^3 + \dots + dY^3 \in V^{\text{irred}}(\mathbb{Z})$.

Then, $a \neq 0$ (since f is irreducible, hence not divisible by X).

$\Rightarrow |a| \geq 1$ (since $a \in \mathbb{Z}$).

Let $V^{|a| \geq 1} = \{ f = aX^3 + \dots \in V \mid |a| \geq 1 \}$.

Let $(\mathcal{F}_T^{GL})' = \mathcal{F}_T^{GL} \cap \{ M \in GL_2(\mathbb{R}) \mid Mf_1 \text{ or } Mf_2 \in V^{|a| \geq 1}(\mathbb{R}) \}$.

Let $\mathcal{F}_T^{!} = (\mathcal{F}_T^{GL})' \cdot \mathcal{F}^{\mathbb{R}}$ as before.

$$\Rightarrow N(T) = \#\left(\mathcal{F}_T \cap V^{\text{irred, max}}(\mathbb{Z})\right)$$

$$= \#\left(\mathcal{F}_T' \cap V^{\text{irred, max}}(\mathbb{Z})\right).$$

Step 4: For any full lattice $\Lambda \subset V(\mathbb{R}) \cong \mathbb{R}^4$, we have $\#(\mathcal{F}_T' \cap \Lambda) \sim \dfrac{\mathrm{vol}(\mathcal{F}_T)}{\mathrm{covol}(\Lambda)}$ for $T \to \infty$.

As the fund. dom. for $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$, use the convolution

$\mathcal{F}^{SL} :=$ (Siegel's fund. dom.) $*$ (subset A of $SL_2(\mathbb{R})$ of volume 1 such that $\partial((0,1] \cdot A) \subset GL_2(\mathbb{R})$ is Lipschitz).

As when we computed the volume of a fund. dom. for $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$, you can apply Widmer's theorem and bound the error term (since we've cut off the cusp!).

Also, $\mathrm{vol}(\mathcal{F}_T') \sim \mathrm{vol}(\mathcal{F}_T)$ for $T \to \infty$.

("fraction of volume in cusp $\to 0$").

<u>Note</u>: This implies that

$$N(T) = \#\left(\mathcal{F}_T' \cap V^{\text{irred, max}}(\mathbb{Z})\right)$$

$$\leq \#(\mathcal{F}_T' \cap \underbrace{V(\mathbb{Z})}_{\cong \mathbb{Z}^4 \text{ full lattice of covolume } 1})$$

$$\sim \mathrm{vol}(\mathcal{F}_T) = \frac{1}{3}\zeta(2) \cdot T.$$

To get the correct constant, we'll use a sieve.

$\dfrac{1}{3\zeta(3)}$

Step 5: $V^{max}(\mathbb{Z}_p)$ is a compact open subset of $V(\mathbb{Z}_p) \cong \mathbb{Z}_p^3$ of volume $(1-p^{-3})(1-p^{-2})$.

Recall the bij.

$$GL_2(\mathbb{Z}_p)\backslash V^{max}(\mathbb{Z}_p) \longleftrightarrow \{\text{nondeg. cubic ext. } L \text{ of } \mathbb{Q}_p\}$$

$$\mathrm{Stab}_{GL_2(\mathbb{Z}_p)}(f) \cong \mathrm{Aut}_{\mathbb{Q}_p}(L)$$

$$(\mathrm{disc}\,(f)) = D_{L|\mathbb{Q}_p}$$

For any $f \in V^{max}(\mathbb{Z}_p)$ corr. to L, consider the map

$$\eta_f : \underset{\subset \mathbb{Z}_p^4}{GL_2(\mathbb{Z}_p)} \to \underset{\mathbb{Z}_p^4 \cup}{V^{max}(\mathbb{Z}_p)}$$
$$M \mapsto Mf$$

Its abs. Jac. det. at any M is $|\mathrm{disc}(f)| = |D_{L|\mathbb{Q}_p}|$.

Any element of the image of η_f has exactly $\#\mathrm{Stab}_{GL_2(\mathbb{Z}_p)}(f) = \#\mathrm{Aut}(L)$ preimages.

$\Rightarrow$ By change of variables, the image (= the orbit $GL_2(\mathbb{Z}_p)\cdot f$) has volume

$$\mathrm{vol}(GL_2(\mathbb{Z}_p)\cdot f \text{ as a set})$$

$$= \frac{1}{\# \mathrm{Aut}(L)} \cdot \int_{GL_2(\mathbb{Z}_p)} |D_{L|\mathbb{Q}_p}| \, d^+ M$$

$$= \frac{|D_{L|\mathbb{Q}_p}|}{\# \mathrm{Aut}(L)} \cdot \mathrm{vol}^+(GL_2(\mathbb{Z}_p))$$

$$= \frac{|D_{L|\mathbb{Q}_p}|}{\# \mathrm{Aut}(L)} \cdot (1-p^{-2})(1-p^{-1})$$

Since $V^{max}(\mathbb{Z}_p) = \bigsqcup_L (\text{orbit corr. to } L)$,

we get

$$\mathrm{vol}(V^{max}(\mathbb{Z}_p)) = \sum_L \frac{|D_{L|\mathbb{Q}_p}|}{\# \mathrm{Aut}(L)} \cdot (1-p^{-2})(1-p^{-1})$$

$$= (1+p^{-1}+p^{-2}) \cdot (1-p^{-2})(1-p^{-1})$$

Bhargava, Kedlaya's mass formula

$$= (1-p^{-3})(1-p^{-2}).$$