(Davenport, Heilbronn: On the density of disc. of cubic fields I + II

Bhargava, Shankar, Tsimerman:
On the Davenport-Heilbronn theorem and second-order terms.)

Thm Define a commutative R-bilinear mult. op. on a free R-module $S = \langle 1, \omega_1, \omega_2 \rangle$ as follows, with $a, b, c, d, n, m, l \in R$:

$$\omega_1 \omega_2 = n$$
$$\omega_1^2 = m - b\omega_1 + a\omega_2$$
$$\omega_2^2 = l - d\omega_1 + c\omega_2$$
$$\begin{pmatrix} 1 \cdot 1 = 1 \\ 1 \cdot \omega_1 = \omega_1 \\ 1 \cdot \omega_2 = \omega_2 \end{pmatrix}$$

This mult. op. is associative (so we obtain a cubic ext. S of R) if and only if

$$n = -ad, \quad m = -ac, \quad l = -bd.$$

Pf associative $\Leftrightarrow$ $\omega_1(\omega_2^2) = (\omega_1\omega_2)\omega_2$ and $(\omega_1^2)\omega_2 = \omega_1(\omega_1\omega_2)$

$$\underbrace{\omega_1(\omega_2^2)}_{= l\omega_1 - dm + bd\omega_1 - ad\omega_2 + cn} = \underbrace{(\omega_1\omega_2)\omega_2}_{= n\omega_2}$$

...

$\Updownarrow$

$-dm + cn = 0$ and $l + bd = 0$ and $-ad = n$ $\square$

__Cor__ Consider the set of $(S, (\theta_1, \theta_2))$, where S is a cubic ext. of R and (θ_1, θ_2) is a basis of S/R.
Identify $(S, (\theta_1, \theta_2))$ with $(S', (\theta_1', \theta_2'))$ if there is an isom. $S \to S'$ of R-alg. that sends θ_1 to θ_1' and θ_2 to θ_2'. We get a bijection

$$\{(S, (\theta_1, \theta_2))\}/\cong \longleftrightarrow V(R)$$

$$(S, (\theta_1, \theta_2)) \longmapsto f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

with a, b, c, d as in the prev. Thm.

__Thm__ With S, θ_1, θ_2, f as above, let $\varphi_{\theta_1, \theta_2} : S/R \to R$ be the composition of

$$S/R \longrightarrow \Lambda^2(S/R)$$

$$[\alpha] \longmapsto \underbrace{[\alpha] \wedge [\alpha^2]}$$

indep. of the rep. α:

$$[\alpha + r] \wedge [(\alpha + r)^2]$$

$$= [\alpha] \wedge [\alpha^2 + 2\alpha r \cancel{+ r^2}]$$

$$= ([\alpha] \wedge [\alpha^2]) + \cancel{(2r[\alpha] \wedge [\alpha])}$$

and $\Lambda^2(S/R) \longrightarrow R$

$$\theta_1 \wedge \theta_2 \longmapsto 1 .$$

We have $f(X,Y) = \varphi_{\theta_1,\theta_2}([X\theta_1 + Y\theta_2])$.

Pf Let $\alpha = X\theta_1 + Y\theta_2$.

$\Rightarrow \alpha^2 \equiv -(bX^2 + dY^2)\theta_1 + (aX^2 + cY^2)\theta_2 \mod R$

$\Rightarrow [\alpha] \wedge [\alpha^2] = f(X,Y)(\theta_1 \wedge \theta_2)$. $\square$

Lemma The (transitive) action of $GL_2(R)$ on the set of bases (θ_1, θ_2) of S/R (for fixed S) corresponds to the action of $GL_2(R)$ on $V(R)$.

Pf This follows from the previous Thm.

$$(Mf)(v) = \frac{f(M^T v)}{\det(M)} \quad \begin{array}{l} \leftarrow \text{"from the first map"} \\ \leftarrow \text{"from the second map"} \end{array}$$

$\square$

Cor We get a bij.

$$\{\text{cubic ext.s of } R\} \longleftrightarrow GL_2(R) \backslash V(R).$$

Cor Let S corr. to $f \in V(R)$. Then,

$$\mathrm{Stab}_{GL_2(R)}(f) \cong \mathrm{Aut}_R(S).$$

Pf aut. of S

$\|$

R-lin. map $S \to S$ fixing $1 \in S$ and commuting with mult.

$\|$

change of basis $(1, \omega_1, \omega_2)$ that fixes a, b, c, d

$\|$

change of bases (θ_1, θ_2) that fixes a, b, c, d

($\Leftrightarrow$ fixes f)

$\|$

el. of $GL_2(R)$ that fixes f.

$\square$

Exe Consider the triv. eset. $L = K \times K \times K$ of K.

Take $\omega_1 = (1,0,0)$, $\omega_2 = (0,1,0)$. $(1 = (1,1,1))$

This corresponds to

$$f(X,Y) = X^2Y + XY^2 = XY(X+Y).$$

$$\mathrm{Stab}_{GL_2(K)}(f) \cong \mathrm{Aut}_K(L) \cong S_3.$$

Lemma Let $f(X,Y) = aX^3 + bX^2Y + cXY^2 + dY^3 \in V(K)$ with $a \neq 0$. Then, the corr. cubic eset. of K

$$L \cong K[X]/(f(X,1)).$$

Pf The isom. is given by $\omega_1 \mapsto aX$
$\omega_2 \mapsto aX^2 + bX + c$. $\square$

Warning This works only over fields!

Cor The cubic ext. L of K corr. to $f \in \mathcal{V}(K)$ is an int. dom. if and only if f is irreducible.

Pf If $a \neq 0$, then $L \cong K[X]/(f(X,1))$ is an int. dom.

$\iff f(X,1)$ irred.

$\iff f(X,Y)$ irred.

If $a = 0$, then $\omega_1 \omega_2 = 0$ ($\Rightarrow$ not int. dom.)

and $f(X,Y) = Y(bX^2 + cXY + dY^2)$

is not irred.

$\square$

Rmk Let $f \in \mathcal{V}(K)$ with $\operatorname{disc}(f) \neq 0$ corr. to the nondeg. cubic ext. L of K. Then, $\operatorname{Gal}(K^{sep}|K)$ acts on the 3 roots $r_1, r_2, r_3 \in \mathbb{P}^1(K^{sep})$ of f exactly like it acts on the three K-alg. hom. $\rho_1, \rho_2, \rho_3 : L \to K^{sep}$ (by right composition).

Thm Let R be a PID. If S corr. to $f \in V(R)$, then

$$\mathrm{disc}(S) = (\mathrm{disc}(f)).$$

Pf just compute... "$\square$"

Maximal extensions

Def We call a nondegenerate deg. n ext. S of a Dedekind dom. R with field of fractions K maximal if S is the int. closure of R in the nondeg. deg. n ext. $S \otimes_R K$ of K.

We call it maximal at a prime $\mathfrak{q}$ of R if the nondeg. deg. n ext. $S \otimes_R R_{\mathfrak{q}}$ of $R_{\mathfrak{q}}$ (completion of R at $\mathfrak{q}$) is maximal.

Rmk A nondeg. deg. n ext. S of R is max. if and only if there is no deg. n ext. $S' \supsetneq S$ of R.

Key facts

a) Every nondeg. deg. n ext. L of K corresponds to exactly one max. deg. n ext. S of R,

b) rel. disc. $D_{L|K} = \mathrm{disc}(S|R)$

c) $\mathrm{Aut}_K(L) = \mathrm{Aut}_R(S)$

d) Maximality is a local cond. (cf. next page)

Thm Let S be a nondeg. deg. n eset. of a Dedekind dom. R. Then:

S maximal $\iff$ S maximal at every $\mathfrak{p}$

Pf "$\Leftarrow$" $S = \{x \in \underbrace{S \otimes K}_{L} \mid x \in S \otimes R_{\mathfrak{p}} \ \forall \mathfrak{p}\}$

$$= \text{"}\bigcap_{\mathfrak{p}} (S \otimes R_{\mathfrak{p}})\text{"}$$

+ Rmk

"$\Rightarrow$" R is dense in $R_{\mathfrak{p}}$

$\Rightarrow$ $S = S \otimes_R R$ is dense in $S \otimes R_{\mathfrak{p}}$

+ Rmk $\square$

Rmk

$$\begin{array}{ccc} \text{max.} & \iff & \text{max. at every } \mathfrak{p} \\ \Uparrow & & \Uparrow \\ \text{disc. sqfree} & \iff & \text{disc. sqfree at every } \mathfrak{p} \\ & & (\mathfrak{p}^2 \nmid \text{disc}) \end{array}$$

For cubic ext., denote by $V^{max}(R)$ the set of $f \in V_{disc \neq 0}(R)$ corr. to max. ext. S of R.

Big goal

<u>Thm</u> $$N(T) := \sum_{\substack{\text{deg. 3 \underline{field}} \\ \text{ext. } L|\mathbb{Q} \\ \text{with } |D_L| \leq T}} \underbrace{\frac{1}{\# \operatorname{Aut}(L)}}_{=1 \text{ ~~for~~ } 100\% \text{ of } L} \sim \frac{1}{3\zeta(3)} \cdot T$$

for $T \to \infty$.

In part. $\#\{\text{deg. 3 field ext. } L|\mathbb{Q} \text{ with } |D_L| \leq T\} \sim \frac{1}{3\zeta(3)} \cdot T$ for $T \to \infty$.