Thm Let K be a nonarch. local field. Consider the nondeg. deg. n étale. $L|K$. We have

$$a_n := \sum_{\substack{L \text{ up to} \\ \text{isom.}}} \frac{|D_{L|K}|}{\# \mathrm{Aut}_K(L)} = \sum_{k=0}^{n} \frac{P(n,k)}{q^{n-k}},$$

where $P(n,k)$ is the number of partitions of the integer n into k positive summands (modulo order).

(Bhargava: Mass formulae for étale. of local fields (Thm 1.1)
Kedlaya: Mass formulas for local Galois repr. (— " —))

Ex $a_0 = 1$ $\quad (L = 1)$

$a_1 = 1$ $\quad (L = K)$

$a_2 = 1 + q^{-1}$ (if $2 \nmid q$, then the étale. are
$L = K \times K, K(\sqrt{a}), K(\sqrt{\pi}), K(\sqrt{a\pi})$
where $a \in \mathcal{O}_K^\times$ is a quadr. nonresidue,
all have 2 automorphisms,
the disc. are $1, 1, q, q$)

$a_3 = 1 + q^{-1} + q^{-2}$

$a_4 = 1 + q^{-1} + 2q^{-2} + q^{-3}$

Pf We can write $L = L_1 \times \dots \times L_r$ with $D_{L|K} = D_{L_1|K} \cdots D_{L_r|K}$ and $n = [L_1 : K] + \dots + [L_r : K]$.

Consider the permutation action of S_r on the set of tuples $(L_1, \dots, L_r)$.

isom. classes

$$\# \mathrm{Aut}(L) = \# \mathrm{Aut}(L_1) \cdots \# \mathrm{Aut}(L_r) \cdot \# \mathrm{Stab}_{S_r}((L_1, \dots, L_r))$$

$$\mathrm{Aut}(L) = (\mathrm{Aut}(L_1) \times \dots \times \mathrm{Aut}(L_r)) \rtimes \mathrm{Stab}_{S_r}((L_1, \dots, L_r)).$$

$$\Rightarrow a_n = \sum_{L \text{ deg. } n} \frac{|D_{L|K}|}{\# \mathrm{Aut}(L)}$$

$$= \sum_{r \geq 0} \sum_{\substack{S_r\text{-orbit} \\ [(L_1, \dots, L_r)] \\ \text{with } n = \sum_{i=1}^{r} [L_i : K]}} \frac{|D_{L_1|K}| \cdots |D_{L_r|K}|}{\# \mathrm{Aut}(L_1) \cdots \# \mathrm{Aut}(L_r)} \cdot \frac{1}{\# \mathrm{Stab}_{S_r}((L_1, \dots))}$$

$$\underset{\substack{\uparrow \\ \text{orbit-stab. thm.}}}{=} \sum_{r \geq 0} \frac{1}{r!} \sum_{\substack{(L_1, \dots, L_r) \\ \dots = n}} \frac{|D_{L_1|K}| \cdots |D_{L_r|K}|}{\# \mathrm{Aut}(L_1) \cdots \# \mathrm{Aut}(L_r)}$$

Use generating function:

$$\sum_{n\geq 0} a_n (qX)^n = \sum_{r\geq 0} \frac{1}{r!}\left(\sum_{\substack{L \text{ field ext.} \\ (\text{up to } \cong)}} \frac{|D_{L|K}|}{\#\text{Aut}(L)} \cdot (qX)^{[L:K]} \right)^r$$

$$= \sum_{r\geq 0} \frac{1}{r!}\left(\sum_{e,f\geq 1} \frac{1}{f q^{ef-f}} \cdot (qX)^{ef} \right)^r$$

$$= \exp\left(\sum_{e,f\geq 1} \underbrace{\frac{1}{f q^{ef-f}} \cdot (qX)^{ef}}_{\frac{q^f \cdot X^{ef}}{f} = \frac{(qX^e)^f}{f}} \right)$$

$$= \exp\left(\sum_{e\geq 1} \log \frac{1}{1-qX^e} \right)$$

$$= \prod_{e\geq 1} \frac{1}{1-qX^e} = \prod_{e\geq 1} \sum_{t\geq 0} (qX^e)^t$$

$$= \sum_{t_1,t_2,\ldots\geq 0} q^{t_1+t_2+\ldots} X^{1\cdot t_1+2\cdot t_2+3\cdot t_3+\ldots}$$

$$\overset{?}{=} \sum_{n\geq 0} \sum_{k\geq 0} P(n,k)\, q^k X^n \implies a_n q^n = \sum_{k\geq 0} P(n,k)\, q^k.$$

write a part. of n into k summands

$n = 1\cdot t_1 + 2\cdot t_2 + \ldots$

$k = t_1 + t_2 + \ldots$

$\square$

Global fields

Binary cubic forms

Let R be an int. dom. with field of fractions K.
Let $V(R)$ be the set of binary cubic forms with coeff. in R:
pol. $f(X,Y) = aX^3 + bX^2Y + cXY^2 + dY^3 \in R[X]$

The discriminant is

$$\mathrm{disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$
$$\underset{\text{if } a \neq 0}{=} \mathrm{disc}(f(X,1))$$
$$\underset{\text{if } d \neq 0}{=} \mathrm{disc}(f(1,X)).$$

Let $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(R)$ act on $f \in V(R)$
by $(Mf)(v) = \dfrac{f(M^T v)}{\det(M)}$, $\quad \left(v = \begin{pmatrix} X \\ Y \end{pmatrix}\right)$

i.e. $(Mf)(X,Y) = \dfrac{f(pX + rY, qX + sY)}{\det(M)}$

Exe $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} f = \lambda \cdot f$

Lemma 1 a) $\operatorname{disc}(Mf) = \det(M)^2 \cdot \operatorname{disc}(f)$

b) The linear map $\varphi_M : V(K) \longrightarrow V(K)$
$$f \longmapsto Mf$$
has determinant $\det(\varphi_M) = \det(M)^2$.

Pf $GL_2(K)$ is gen. by matrices of the form
$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}.$$
$\Rightarrow$ Suffices to check the claims for matrices M of these forms.

a) $\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} f\right)(x,1) = f(x+t,1)$

$\Rightarrow \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} f\right)(x,1)$ and $f(x,1)$ have same leading coeff.

and roots are shifted by t.

$\Rightarrow$ same disc.

...

$\square$

Lemma 2 Let $f \in V(K)$. The abs. value of the Jacobian determinant of

$$\eta_f : GL_2(K) \longrightarrow V(K)$$
$$M \mapsto Mf$$

at $M \in GL_2(K)$ w.r.t.

the standard 4-form on $M_2(K) \cong K^4$ ($\rightsquigarrow$ Lebesgue measure)
$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \Leftrightarrow (p, \dots)$

and the standard 4-form on $V(K) \cong K^4$
$ax^3 + \dots \Leftrightarrow (a, \dots)$

is $|\det \mathrm{Jac}(\eta_f)(M)| = |\mathrm{disc}(f)|$.

Pf Let $\rho_M : GL_2(K) \to GL_2(K)$ be the right mult. by M map.

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\Rightarrow \eta_{Mf} = \eta_f \circ \rho_M$

$\underset{\substack{\uparrow \\ \text{chain rule}}}{\Rightarrow}$ $\mathrm{Jac}(\eta_{Mf})(I) = \mathrm{Jac}(\eta_f)(M) \cdot \mathrm{Jac}(\rho_M)(I)$

$$\Rightarrow \underbrace{|\det \text{''} \quad |}_{\overset{!}{=} |\mathrm{disc}(Mf)|} = \underbrace{|\det \text{''} \quad |}_{\overset{?}{=} |\mathrm{disc}(f)|} \cdot \underbrace{|\det \text{''} \quad |}_{|\det(M)|^2}$$

$\Rightarrow$ By Lemma 1a, it suffices to check the claim for $M = I$ and all $f \in V(R)$.

$$\frac{\partial}{\partial t}\left(\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} f\right)(x,y)\right)\Big|_{t=0} = bx^3 + 2cx^2y + 3dxy^2$$

$$\frac{\partial}{\partial t}\left(\left(\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix} f\right)(x,y)\right)\Big|_{t=0} = 3ax^2y + 2bxy^2 + cy^3$$

$$\frac{\partial}{\partial t}\left(\left(\begin{pmatrix}1+t & 0\\ 0 & 1\end{pmatrix} f\right)(x,y)\right)\Big|_{t=0} = 2ax^3 + bx^2y - dy^3$$

$$\frac{\partial}{\partial t}\left(\left(\begin{pmatrix}1 & 0\\ 0 & 1+t\end{pmatrix} f\right)(x,y)\right)\Big|_{t=0} = -ax^3 + cxy^2 + 2dy^3.$$

$$\Rightarrow \left|\det \operatorname{Jac}(\eta_f)(I)\right| = \left|\det\begin{pmatrix} b & 2c & 3d & 0\\ 0 & 3a & 2b & c\\ 2a & b & 0 & -d\\ -a & 0 & c & 2d\end{pmatrix}\right| = |\operatorname{disc}(f)|,$$

$\square$

3 points in $\mathbb{P}^1$

Let $V_{disc \neq 0} = \{ f \in V \mid disc(f) \neq 0 \}$.

The following bij. is helpful in understanding the action of $GL_2(\bar{K})$ on $V_{disc \neq 0}(\bar{K})$:

$$V_{disc \neq 0}(\bar{K}) / \bar{K}^\times \longleftrightarrow \{\text{sets } S \text{ of three (dist.) pts. on } \mathbb{P}^1(\bar{K})\}$$

$$[f] \longmapsto \text{roots } [x:y] \in \mathbb{P}^1(\bar{K}) \text{ of}$$

$$\left[\prod_{i=1}^{3} (b_i X - a_i Y)\right] \longleftarrow \{ (a_i : b_i) \mid i = 1, 2, 3 \}$$

Let $\underset{\substack{\Psi \\ [M]}}{PGL_2(\bar{K})}$ act on $\underset{\substack{\Psi \\ [x:y]}}{\mathbb{P}^1(\bar{K})}$ by $M(x:y] = [x':y']$

with $\begin{pmatrix} x' \\ y' \end{pmatrix} = (M^T)^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$, This makes the bijection $PGL_2(\bar{K})$-equivariant.

It turns out that $PGL_2(\bar{K})$ acts simply transitively on the set of (ordered!) tuples (P_1, P_2, P_3) of three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1(\bar{K})$.

$$\Rightarrow \mathrm{Stab}_{PGL_2(\bar{K})}([f]) = \mathrm{Stab}_{PGL_2(\bar{K})}(\text{set of roots of } f) \cong S_3$$

perm. of the roots

Since $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} f = \lambda \cdot f$, it follows that:

Lemma 3 $\quad \mathrm{Stab}_{GL_2(\bar{K})}(f) \cong S_3 \quad$ (for $f \in V(K)$)

$$\begin{array}{c} \mathrm{Stab}_{GL_2(\bar{K})}(f) \\ \cup\!| \\ \mathrm{Stab}_{GL_2(K)}(f) \end{array}$$

Cubic extensions

Consider a cubic (= degree 3) ext. S of a PID R with field of fractions K.

Lemma $\quad S$ has an R-basis of the form $(1, \omega_1, \omega_2)$.

(In part., S/R is a free R-mod. of rank 2.)

Pf $\quad$ Since S is an R-lattice of rank 3 and R is a PID, $\quad S$ is free of rank 3.

Consider the embedding

$$\begin{array}{ccc} S & \hookrightarrow & S \otimes_R K \\ \uparrow & & \uparrow \\ R & \hookrightarrow & K \end{array}.$$

Every $x \in S$ is integral over R (it's a root of the char. pol. of the mult. by x map $S \to S$).

$\Rightarrow S_{\cap} K = R$

$\Rightarrow 1 \in S$ is a primitive vector in the lattice S.

$\Rightarrow$ S has a basis of the form $(1, \omega_1, \omega_2)$. $\square$

<u>Rmk</u> Let (θ_1, θ_2) be a basis of the R-module S/R. Then, there is a unique basis $(1, \omega_1, \omega_2)$ of S with $\omega_i \equiv \theta_i \mod R$ such that $\omega_1 \omega_2 \in R$.

<u>Pf</u> Take any $\omega_1' \equiv \theta_1$, $\omega_2' \equiv \theta_2 \mod R$. Then, $(1, \omega_1', \omega_2')$ is a basis of S.

$\Rightarrow$ We can write

$$\omega_1' \omega_2' = n \cdot 1 + p \cdot \omega_1' + q \cdot \omega_2' \text{ with } n, p, q \in R.$$

Write $\omega_1 = \omega_1' + \delta_1$, $\omega_2 = \omega_2' + \delta_2$ with $\delta_1, \delta_2 \in R$.

$$\Rightarrow \omega_1 \omega_2 = (n + \delta_1 \delta_2) \cdot 1 + (p + \delta_2) \cdot \omega_1' + (q + \delta_1) \omega_2' \in R$$

if and only if $p + \delta_2 = q + \delta_1 = 0$. $\square$

(Davenport, Heilbronn: On the density of disc. of cubic fields I + II

Bhargava, Shankar, Tsimerman:
On the Davenport-Heilbronn theorem and second-order terms.)