Thm Let $\mathcal{F}$ be a measurable fund. dom. for $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$.

For $T>0$, consider the fund. dom. $\mathcal{F}_T = (0,T]\cdot\mathcal{F}$

for $SL_n(\mathbb{Z})\backslash GL_n^{T^n}(\mathbb{R})$. ($0<\det\leq T^n$)

Then, (Lebesgue!)

$$\#(\mathcal{F}_T \cap M_n(\mathbb{Z})) \sim \mathrm{vol}^+(\mathcal{F}_T) \quad \text{for } T\to\infty.$$

Pf Both sides are indep. of the choice of fund. dom. $\mathcal{F}_T$. (The action of $SL_n(\mathbb{R})$ preserves the Lebesgue measure.)

Assume w.l.o.g. that $\mathrm{supp}(\mathcal{F}) \subset \widetilde{\mathcal{F}}_{Siegel}^{SL_n(\mathbb{R})} = N' A'_1 K_1$.

[Now, use convolution to make $\mathcal{F}$ nicer!]

Fix any subset $S\subset SL_n(\mathbb{R})$ of volume 1 whose boundary is Lipschitz.

$\Rightarrow$ $\mathcal{F} * S$ is also a fund. dom. for $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$,

$(\mathcal{F}*S)_T = (0,T]\cdot(\mathcal{F}*S)$ is also a fund. dom. for $SL_n(\mathbb{Z})\backslash GL_n^{T^n}(\mathbb{R})$,

with $\mathrm{vol}^+((\mathcal{F}*S)_T) = \mathrm{vol}^+(\mathcal{F}_T)$.

$\Rightarrow$ It suffices to prove

$$\#((\mathcal{F}*S)_T \cap M_n(\mathbb{Z})) \sim \mathrm{vol}^+(\mathcal{F}_T) \quad \text{for } T\to\infty.$$

But $$LHS = \int_{\mathcal{F}} \underbrace{\#(((0,T]\cdot gS)\cap M_n(\mathbb{Z}))}_{f(g)}\,dg = \int_{\mathcal{F}} f(g)\,dg.$$

Now, we want to apply Widmer's Thm. to the integrand.

Write $g = nak$ with $n\in N'$, $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}\in A'_1$,

$k\in K_1 = SO_n(\mathbb{R})$. The set gS could be narrow and long if a_1 is small and a_n is large!

$\rightsquigarrow$ It'll be better to rescale the lattice $M_n(\mathbb{Z})$ than the set S.

$$f(g) = \#((0,T]\cdot \underbrace{g}_{nak} S \cap M_n(\mathbb{Z})) = \#((0,T]\cdot kS \cap (na)^{-1} M_n(\mathbb{Z}))$$

End of lecture 13

Since K_1 is compact, $\partial(k \cdot S)$ is $(O_s(1), O_s(1))$-Lipschitz.

$\Rightarrow \partial((0,T] \cdot k S)$ is $(O_s(1), O_s(T))$-Lipschitz.

Also, $kS \subset M_n(\mathbb{R})$ is contained in a ball of radius $O_s(1)$,

so $(0,T] \cdot kS$ is contained in a ball of radius $O_s(T)$.

Since $N' \subset SL_n(\mathbb{R})$ is compact and any $a \in A'$ satisfies $a_1 \ll \dots \ll a_n$, the previous lemma shows that the succ. min. $\lambda_1 \leq \dots \leq \lambda_n$ of $\Lambda = (n\,a)^{-1}\mathbb{Z}^n$ satisfy $\lambda_i \asymp a_{n+1-i}^{-1}$.

Note that $(n\,a)^{-1} M_n(\mathbb{Z}) \equiv \Lambda^n$ consists of the matrices whose columns lie in Λ.

Λ^n has the same succ. min. as Λ, with each λ_i occuring n times. [could apply Widmer for $f(g)$, but the integral of the error term would be ∞!]

If $f(g) = \#(\underbrace{(0,T] \cdot kS}_{\subset GL_n(\mathbb{R})} \cap (n\,a)^{-1} M_n(\mathbb{Z})) \neq 0$, there must be n linearly independent vectors in Λ of length $O_s(T)$.

$$\Rightarrow T \gg_s \lambda_n \asymp a_1^{-1} \gg \dots \gg a_n^{-1}.$$

$\rightsquigarrow$ Cut off cusp: Let $\mathcal{F}^{(T)} = \mathcal{F} \cap \{ g = n\,a\,k \mid a_1^{-1} \ll_s T \}$.

$$\Rightarrow LHS = \int_{\mathcal{F}^{(T)}} f(g)\,dg.$$

$\mathcal{F}^{(T)} \to \mathcal{F}$ (monotonically) for $T \to \infty$

$$RHS = vol^+(\mathcal{F}_T) \equiv vol^+((0,T] \cdot \mathcal{F}) \sim vol^+((0,T] \cdot \mathcal{F}^{(T)})$$

Let $g = nak \in \operatorname{supp}(\mathcal{F}^{(T)})$. By Widmer's theorem,

$$f(g) = \frac{\mathrm{vol}^+((0,T]\cdot kS)}{\mathrm{covol}(\Lambda^n)} + \sum_{l=0}^{n^2-1} O_S\left(\frac{T^l}{\text{prod. of } l \text{ smallest succ. min. of } \Lambda^n}\right)$$

$xa_1^{-1},\dots,xa_n^{-1}$, each n times

$g, k \in SL_n(\mathbb{R})$ preserve Lebesgue measure

$T \gg a_1^{-1} \gg \dots \gg a_n^{-1}$ and $\prod a_i = 1$

$$= \frac{\mathrm{vol}^+((0,T]\cdot S)}{1} + O_S\left(\frac{T^{n^2-1}\cdot a_1^{-1}}{1}\right)$$

$$\Rightarrow \int_{\mathcal{F}^{(T)}} f(g)\,dg = \int_{\mathcal{F}^{(T)}} \left(\mathrm{vol}^+((0,T]\cdot S) + O_S\left(\frac{T^{n^2-1}}{a_1}\right)\right)dg$$

main term $= \int_{\mathcal{F}^{(T)}} \mathrm{vol}^+((0,T]\cdot S)\,dg = \mathrm{vol}^\times(\mathcal{F}^{(T)})\cdot \mathrm{vol}^+((0,T]\cdot S)$

$= \frac{T^n}{n}\mathrm{vol}^\times(\mathcal{F}^{(T)})\cdot \mathrm{vol}^\times(S) = \mathrm{vol}^+((0,T]\cdot \mathcal{F}^{(T)})\cdot \underbrace{\mathrm{vol}^\times(S)}_{1} \times T^{n^2}$ ✓

[We did this computation last time.]

$$\frac{\text{error term}}{\text{main term}} \ll \int_{\mathcal{F}^{(T)}} \frac{1}{Ta_1}\,dg \ll \int_{\operatorname{supp}(\mathcal{F}^{(T)})} \frac{1}{Ta_1}\,dg$$

$$\le \int_{N'A_1'K_1} \frac{1}{Ta_1}\,dg = \int_{N'}\int_{A_1'}\int_{K_1} \frac{1}{Ta_1}\,dk\,da\,dn$$

(AS, 107)

$$\overset{\times}{\ll} \int_{N'} \int_{\left[\frac{\sqrt{3}}{2},\infty\right]^{n-1} \subset B = (\mathbb{R}^{>0})^{n-1}} \int_{K_1} \frac{b_1^{n-1} \cdots b_{n-1}}{T} \frac{d^\times k \, d^\times b \, d^\times n}{\prod_{i=1}^{n-1} b_i^{ni(n-i)}} \ll \frac{1}{T} \xrightarrow{T\to\infty} 0 \checkmark$$

Formula for Haar measure on $SL_n(\mathbb{R})$,

$$\frac{a_{i+1}}{a_i} = b_i^n,$$

$$a_1 = \frac{1}{b_1^{n-1} \cdots b_1}$$

$\square$

p-adic Haar measure.

Let k be a (non-arch.) local field with ring of integers $\mathcal{O}_k$, prime $\mathfrak{q} = (\pi)$ (valuation v), residue field $\kappa = k/\mathfrak{q}$ of order q, norm $|x| = q^{-v(x)}$ for $x \in k^\times$.

We normalize the Haar measure $dx = d^+x$ on k by $\mathrm{vol}^+(\mathcal{O}_k) = 1$ (compact).
⇝ The restriction to $\mathcal{O}_k$ is a probability measure.

__Rmk__ For $\lambda \in k^\times$, we have $d(\lambda x) = |\lambda|\, dx$.

__Pf__ $d(\lambda x)$ is also a Haar measure on k.
By uniqueness of Haar measures, it suffices to show that

$$\mathrm{vol}(\lambda \mathcal{O}_k) = |\lambda| \, \mathrm{vol}(\mathcal{O}_k).$$

Since $k^\times = \mathcal{O}_k^\times \times \pi^{\mathbb{Z}}$, it suffices to prove this for $\lambda \in \mathcal{O}_k^\times$ and $\lambda = \pi$.

For $\lambda \in \mathcal{O}_k^\times$, $\lambda \mathcal{O}_k = \mathcal{O}_k$ and $|\lambda| = 1$.

For $\lambda = \pi$, note that $\mathcal{O}_k$ is the disjoint union of q translates of $\mathfrak{q} = \pi \mathcal{O}_k$ (residue classes), so $\mathrm{vol}(\pi \mathcal{O}_k) = \frac{1}{q}$, and $|\pi| = q^{-1}$.

□

⇝ We get a mult. Haar measure $d^\times x = \frac{dx}{|x|}$ on $k^\times$.

__Rmk__ For $A \subseteq \mathcal{O}_k/\mathfrak{q}^e$, we have

$$\underset{\mathrm{vol}(\{x \in \mathcal{O}_k : (x \bmod \mathfrak{q}^e) \in A\})}{\underset{\shortparallel}{\mathbb{P}((x \bmod \mathfrak{q}^e) \in A \mid x \in \mathcal{O}_k)}} = \mathbb{P}(x \in A \mid x \in \mathcal{O}_k) = \frac{\# A}{q^e}.$$

__Rmk__ Let $S \subset \mathcal{O}_k$ be a set of representatives for the q residue classes.
We can write any $x \in \mathcal{O}_k$ uniquely as $x = \sum_{i=0}^{\infty} \underset{\text{"digits"}}{\underset{\uparrow}{c_i}} \pi^i$ with $c_i \in S$.
⇝ bijection $\mathcal{O}_k \longleftrightarrow \prod_{i=0}^{\infty} S$.
The Haar measure on $\mathcal{O}_k$ is (on Borel sets) the product measure, where we endow S with the uniform probability measure.
[Roll dice for each digit.]

<u>Ese</u> $vol^+(\mathcal{O}_k^\times) = vol^\times(\mathcal{O}_k^\times) = vol^+(\mathcal{O}_k) - vol^+(\mathfrak{p}) = 1 - q^{-1}$

$|x| = 1$ for $x \in \mathcal{O}_k^\times$

$= \mathbb{P}(x \neq 0 \mid x \in \kappa)$

Define Haar measures on $GL_n(k)$, $SL_n(k)$ as over $\mathbb{R}$.

<u>lemma</u> $vol^+(GL_n(\mathcal{O}_k)) = vol^\times(GL_n(\mathcal{O}_k)) = \prod_{i=1}^{n} (1-q^{-i})$

$|\det(g)| = 1$ for $g \in GL_n(\mathcal{O}_k)$

<u>Pf</u> $LHS = \mathbb{P}(g \in GL_n(\kappa) \mid g \in M_n(\kappa))$

$= \mathbb{P}(v_1, \dots, v_n \text{ lin. indep.} \mid v_1, \dots, v_n \in \kappa^n)$

look at col. of g

$= \mathbb{P}(v_1 \neq 0) \cdot \mathbb{P}(v_2 \notin \langle v_1 \rangle \mid v_1 \neq 0) \cdot \ldots \cdot \mathbb{P}(v_n \notin \langle v_1, \dots, v_{n-1} \rangle \mid v_1, \dots, v_{n-1} \text{ lin. indep.})$

$= (1-q^{-n})(1-q \cdot q^{-n}) \cdots (1 - q^{n-1} \cdot q^{-n})$

$= (1-q^{-1})(1-q^{-2}) \cdots (1-q^{-n})$. $\square$

<u>lemma</u> $vol^+(SL_n(\mathcal{O}_k)) = vol^\times(SL_n(\mathcal{O}_k)) = \prod_{i=2}^{n} (1-q^{-i})$

<u>Pf</u> Under the homeomorphism $SL_n(\mathcal{O}_k) \times \mathcal{O}_k^\times \xrightarrow{\sim} GL_n(\mathcal{O}_k)$, $(h, t) \longmapsto h \begin{pmatrix} t & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{pmatrix}$, the Haar measure $d^\times g$ on $GL_n(\mathcal{O}_k)$ pulls back to $d^\times h \, d^\times t$.

$\Rightarrow vol^\times(SL_n(\mathcal{O}_k)) \cdot \underbrace{vol^\times(\mathcal{O}_k^\times)}_{1-q^{-1}} = \underbrace{vol^\times(GL_n(\mathcal{O}_k))}_{\prod_{i=1}^{n}(1-q^{-i})}$. $\square$

<u>Strong approximation + Tamagawa number</u> Let $\mathbb{A} = \mathbb{A}(\mathbb{Q}) = \prod_v' \mathbb{Q}_v = \underset{v=\infty}{\mathbb{R}} \times \prod_p' \mathbb{Q}_p$ be the ring of adeles and $\mathbb{A}_{fin} = \prod_p' \mathbb{Q}_p$.

(AS, 110)

<u>Thm</u> (Strong approx. for $\mathbb{G}_a$ (over $\mathbb{Q}$, away from ∞))
The image of $\mathbb{Q}$ in $\mathbb{A}_{fin}$ is dense (in $\mathbb{A}_{fin}$).

<u>Pf</u> We need to show that every open set $U \overset{\mathbb{A}_{fin}}{\subseteq}$ contains an element of $\mathbb{Q}$. It suffices to show this for basis open sets

$$U = \prod_{p\in S} U_p \times \prod_{p\notin S} \mathbb{Z}_p,$$ where S is a finite set of primes and

$U_p \subseteq \mathbb{Q}_p$ is open. W.l.o.g. $U_p = y_p + p^{e_p}\mathbb{Z}_p$ with $y_p \in \mathbb{Q}_p$, $e_p \in \mathbb{Z}$.
Multiplying by large enough powers of $p \in S$, we can assume $y_p \in \mathbb{Z}_p$, $e_p \geq 0$.
By the <u>CRT</u>, there exists $x \in \mathbb{Z}$ s.t. $x \equiv y_p \bmod p^{e_p}$ $\forall p$.
$\Rightarrow x \in U$. □

<u>Cor</u> For any $y = (y_p)_p \in \mathbb{A}_{fin}$, there is some $x \in \mathbb{Q}$ such that $x + y = (x+y_p)_p \in \prod_p \mathbb{Z}_p$.

<u>Pf</u> $U = \prod_p (\mathbb{Z}_p - y_p)$ is an open subset of $\mathbb{A}_{fin}$.

$$\overset{SA}{\Rightarrow} \exists x \in \mathbb{Q}: \ x \in \mathbb{Z}_p - y_p \ \forall p$$
$$\Downarrow$$
$$x + y_p \in \mathbb{Z}_p$$
□

<u>Cor</u> The set $[0,1) \times \prod_p \mathbb{Z}_p$ is a fund. dom. for $\mathbb{Q}\backslash\mathbb{A}$. Its volume (the <u>Tamagawa number</u> of $\mathbb{G}_a$ over $\mathbb{Q}$) is 1.

<u>Pf</u> By the prev. cor. every $\mathbb{Q}$-orbit contains some $y \in \mathbb{R} \times \prod_p \mathbb{Z}_p$.

$$\Rightarrow \sum_{x\in\mathbb{Q}} \chi_{[0,1)\times\prod_p \mathbb{Z}_p}(x+y) = \sum_{x\in\mathbb{Q}} \chi_{[0,1)}(x+y_\infty) \underbrace{\prod_p \underbrace{\chi_{\mathbb{Z}_p}(x+y)}_{=1 \Leftrightarrow x\in\mathbb{Z}_p}}_{=1 \Leftrightarrow x \in \mathbb{Z}} = \sum_{x\in\mathbb{Z}} \chi_{[0,1)}(x+y_\infty) = 1$$ □

$[0,1)$ is fund. dom. for $\mathbb{Z}\backslash\mathbb{R}$.

$\mathrm{vol}([0,1)) = 1$
$\mathrm{vol}(\mathbb{Z}_p) = 1$.

AS, 111

Thm (Strong approx. for SL_n)

The image of $SL_n(\mathbb{Q})$ in $SL_n(\mathbb{A}_{fin})$ is dense.

Pf By def. of the top. on $SL_n(\mathbb{A}_{fin})$, it suffices to prove that the closure of the image of $SL_n(\mathbb{Q})$ contains $SL_n(\mathbb{Q}_p)$ for every p. For $a \neq b$, consider the subgroup G_{ab} of SL_n consisting of matrices $(m_{ij})_{i,j}$ with $m_{ij} = 1$ for $i = j$, $m_{ij} = 0$ for $i \neq j$ if $(i,j) \neq (a,b)$.

$$m = \begin{array}{cc} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 \quad * & & 1 \end{bmatrix} & \leftarrow a \\ \uparrow & \\ b & \end{array}$$

We have $G_{ab} \cong \mathbb{G}_a$ (i.e. $G_{ab}(R) = R$ for any ring R).

Now, $SL_n(\mathbb{Q}_p)$ is generated by the el. of the subgroups $G_{a,b}(\mathbb{Q}_p)$ for $a \neq b$. $\Rightarrow$ It suffices to prove that the closure of the image of $G_{a,b}(\mathbb{Q}) \subset SL_n(\mathbb{Q})$ contains $G_{a,b}(\mathbb{Q}_p) \subset SL_n(\mathbb{Q}_p)$. This follows from strong approx. for $\mathbb{G}_a$. $\square$

Cor Let F be a fund. dom. for $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$. Then, $F \times \prod_p SL_n(\mathbb{Z}_p)$ is a fund. dom. for $SL_n(\mathbb{Q}) \backslash SL_n(\mathbb{A})$. Its volume (the Tamagawa number of SL_n over $\mathbb{Q}$) is 1.

Pf Fund. dom.: Follows from SA like for $\mathbb{G}_a$.

Volume: $\mathrm{vol}(F) = \zeta(2) \cdots \zeta(n)$

$\mathrm{vol}(SL_n(\mathbb{Z}_p)) = (1-p^{-2}) \cdots (1-p^{-n})$

$\prod \quad \cdots = 1.$ $\square$

Weil's conjecture on Tamagawa numbers (known)

The Tamagawa number of a simply connected simple algebraic group over a number field is 1.

End of lecture 14