Math 286X: Arithmetic Statistics

Spring 2020

Lattices and maximal extensions

Let R be a Dedekind domain with field of fractions K. By a prime $\mathfrak{p}$, we mean a nonzero prime ideal of R. For any prime $\mathfrak{p}$, denote the localization of R at $\mathfrak{p}$ by $R_\mathfrak{p}$, the $\mathfrak{p}$-adic completion of R by $\hat{R}_\mathfrak{p}$, and the $\mathfrak{p}$-adic completion of K by $\hat{K}_\mathfrak{p}$. (In class, we denoted the completions by $R_\mathfrak{p}$ and $K_\mathfrak{p}$.)

A *full R-lattice in K^n* is a finitely generated R-submodule $\Lambda \subseteq K^n$ whose elements span the K-vector space K^n. (This is equivalent to $\Lambda \otimes_R K \cong K^n$. It is also equivalent to the assumption that $aR^n \subseteq \Lambda \subseteq bR^n$ for some $a, b \in K^\times$.) For example, the full R-lattices in K are exactly the (nonzero) fractional ideals of R.

1 Localization

If $T_\mathfrak{p}$ is a subset of $R_\mathfrak{p}^n$ for all $\mathfrak{p}$, we write $\bigcap_\mathfrak{p} T_\mathfrak{p}$ for the set of $x \in K^n$ such that $x \in T_\mathfrak{p}$ for all $\mathfrak{p}$.

Theorem 1 ([Rei03, Theorems 4.21, 4.22]). *Consider the set of full R-lattices Λ in K^n and the set of tuples $(\Lambda_\mathfrak{p})_\mathfrak{p}$, where $\Lambda_\mathfrak{p}$ is a full $R_\mathfrak{p}$-lattice in $K_\mathfrak{p}^n$ for all $\mathfrak{p}$ and $\Lambda_\mathfrak{p} = R_\mathfrak{p}^n$ for almost all $\mathfrak{p}$. We then obtain a bijection*

$$\begin{aligned} \{\Lambda\} &\longleftrightarrow \{(\Lambda_\mathfrak{p})_\mathfrak{p}\}, \\ \Lambda &\longmapsto (\Lambda \otimes_R R_\mathfrak{p})_\mathfrak{p}, \\ \textstyle\bigcap_\mathfrak{p} \Lambda_\mathfrak{p} &\longleftarrow\!\shortmid (\Lambda_\mathfrak{p})_\mathfrak{p}. \end{aligned}$$

For example, a fractional ideal $\mathfrak{a}$ corresponds to the tuple $(\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})})_\mathfrak{p}$ of fractional ideals of the localizations $R_\mathfrak{p}$. Also, note that $\Lambda \otimes_R R_\mathfrak{p} = R_\mathfrak{p} \cdot \Lambda = (R\backslash\mathfrak{p})^{-1} \cdot \Lambda \subset K^n$.

Proof. **"$\rightarrow$ is well-defined"** Let $aR^n \subseteq \Lambda \subseteq bR^n$ as above. For all primes $\mathfrak{p}$ with $v_\mathfrak{p}(a) = v_\mathfrak{p}(b) = 0$ (these are all but finitely many), we have $a, b \in R_\mathfrak{p}^\times$, so $R_\mathfrak{p}^n = aR_\mathfrak{p}^n \subseteq \Lambda \otimes_R R_\mathfrak{p} \subseteq bR_\mathfrak{p}^n = R_\mathfrak{p}^n$. Hence, $\Lambda \otimes_R R_\mathfrak{p} = R_\mathfrak{p}^n$ for almost all $\mathfrak{p}$.

"$\leftarrow$ is well-defined" Note that $\mathfrak{p}^{a_\mathfrak{p}} R_\mathfrak{p}^n \subseteq \Lambda_\mathfrak{p} \subseteq \mathfrak{p}^{b_\mathfrak{p}} R_\mathfrak{p}^n$ for some $a_\mathfrak{p} \leqslant b_\mathfrak{p}$, where $a_\mathfrak{p} = b_\mathfrak{p} = 0$ for almost all $\mathfrak{p}$. Then, $\Lambda = \bigcap_\mathfrak{p} \Lambda_\mathfrak{p}$ satisfies $\prod_\mathfrak{p} \mathfrak{p}^{a_\mathfrak{p}} R^n \subseteq \Lambda \subseteq \prod_\mathfrak{p} \mathfrak{p}^{b_\mathfrak{p}} R^n$, so Λ is a full lattice in K^n.

"$\leftarrow \circ \rightarrow =$ id" Let Λ as above. We need to show that $\bigcap_\mathfrak{p} (\Lambda \otimes_R R_\mathfrak{p}) = \Lambda$. The inclusion $\supseteq$ is clear. To prove $\subseteq$, take any $x \in \bigcap_\mathfrak{p} (\Lambda \otimes_R R_\mathfrak{p})$.

For each $\mathfrak{p}$, there exists some $a_\mathfrak{p} \in R \backslash \mathfrak{p}$ such that $a_\mathfrak{p} x \in \Lambda$. The set I of $b \in R$ such that $bx \in \Lambda$ forms an ideal. Since it isn't contained in any of the nonzero prime ideals $\mathfrak{p}$, we have $I = (1)$, so $x \in \Lambda$.

"$\rightarrow \circ \leftarrow =$ id" Let $(\Lambda_\mathfrak{p})_\mathfrak{p}$ as above, and let $\Lambda = \bigcap_\mathfrak{p} \Lambda_\mathfrak{p}$. Clearly, $\Lambda \otimes_R R_\mathfrak{p} \subseteq \Lambda_\mathfrak{p}$. Again, $\mathfrak{p}^{a_\mathfrak{p}} R_\mathfrak{p}^n \subseteq \Lambda_\mathfrak{p} \subseteq \mathfrak{p}^{b_\mathfrak{p}} R_\mathfrak{p}^n$ for some $a_\mathfrak{p} \leqslant b_\mathfrak{p}$, where $a_\mathfrak{p} = b_\mathfrak{p} = 0$ for almost all $\mathfrak{p}$. Fix a prime $\mathfrak{p}$ and let $x \in \Lambda_\mathfrak{p}$. For any $\mathfrak{q}$, let $c_\mathfrak{q}$ be the smallest nonnegative integer such that $\mathfrak{q}^{c_\mathfrak{q}} x \in \Lambda_\mathfrak{q}$. For almost all $\mathfrak{p}$, we have $c_\mathfrak{q} = 0$. By the Chinese remainder theorem, there exists some $t \in K^\times$ such that $v_\mathfrak{p}(t) = 0$ and such that $v_\mathfrak{q}(t) \geqslant c_\mathfrak{q}$ for all $\mathfrak{q} \neq \mathfrak{p}$. Then, $tx \in \Lambda = \bigcap_\mathfrak{q} \Lambda_\mathfrak{q}$. On the other hand, $t^{-1} \in R_\mathfrak{p}$, so indeed $x = t^{-1} tx \in \Lambda \otimes_R R_\mathfrak{p}$. □

Theorem 2 ([Rei03, Theorem 11.2])**.** *Let S be a degree n extension of R. Then, S is a maximal extension of R if and only if $S \otimes_R R_\mathfrak{p}$ is a maximal extension of $R_\mathfrak{p}$ for all $\mathfrak{p}$.*

Proof. **"$\Rightarrow$"** Let S be a maximal extension of R. For every $\mathfrak{p}$, let $S'_\mathfrak{p} \supseteq S \otimes_R R_\mathfrak{p}$ be a degree n extension of $R_\mathfrak{p}$, and let $S'_\mathfrak{p} = S \otimes_R R_\mathfrak{p}$ for almost all $\mathfrak{p}$. Then, $S \subseteq \bigcap_\mathfrak{p} S'_\mathfrak{p} \subset S \otimes_R K$ is an R-lattice of rank n, which is closed under multiplication. Hence, $\bigcap_\mathfrak{p} S'_\mathfrak{p} \supseteq S$ is a degree n extension of R. Since S is maximal, this implies that $\bigcap_\mathfrak{p} S'_\mathfrak{p} = S$. In follows that $S'_\mathfrak{p} = S \otimes_R R_\mathfrak{p}$ for all $\mathfrak{p}$. Hence, each $S \otimes_R R_\mathfrak{p}$ is a maximal extension of $R_\mathfrak{p}$.

"$\Leftarrow$" Assume that $S \otimes_R R_\mathfrak{p}$ is a maximal extension of $R_\mathfrak{p}$ for all $\mathfrak{p}$. Let $S' \supseteq S$ be a degree n extension of R. Then, $S' \otimes_R R_\mathfrak{p} \supseteq S \otimes_R R_\mathfrak{p}$ is also a degree n extension of $R_\mathfrak{p}$. By maximality, $S' \otimes_R R_\mathfrak{p} = S \otimes_R R_\mathfrak{p}$ for all $\mathfrak{p}$. But then, $S' = S$. Hence, S is a maximal extension of R. □

2 Completion

If $T_\mathfrak{p}$ is a subset of $\hat{R}_\mathfrak{p}^n$ for all $\mathfrak{p}$, we write $\bigcap_\mathfrak{p} T_\mathfrak{p}$ for the set of $x \in K^n$ such that $x \in T_\mathfrak{p}$ for all $\mathfrak{p}$.

Theorem 3 ([Rei03, Theorem 5.3])**.** *Consider the set of full R-lattices Λ in K^n and the set of tuples $(\Lambda_\mathfrak{p})_\mathfrak{p}$, where $\Lambda_\mathfrak{p}$ is a full $\hat{R}_\mathfrak{p}$-lattice in $\hat{K}_\mathfrak{p}^n$ for all $\mathfrak{p}$ and $\Lambda_\mathfrak{p} = \hat{R}_\mathfrak{p}^n$ for almost all $\mathfrak{p}$. We then obtain a bijection*

$$\begin{aligned} \{\Lambda\} &\longleftrightarrow \{(\Lambda_\mathfrak{p})_\mathfrak{p}\}, \\ \Lambda &\longmapsto (\Lambda \otimes_R \hat{R}_\mathfrak{p})_\mathfrak{p}, \\ \textstyle\bigcap_\mathfrak{p} \Lambda_\mathfrak{p} &\longleftarrow\!\shortmid (\Lambda_\mathfrak{p})_\mathfrak{p}. \end{aligned}$$

Proof. Since the $\mathfrak{p}$-adic completion of $R_\mathfrak{p}$ is the same as the $\mathfrak{p}$-adic completion of R, Theorem 1 shows that it suffices to consider local rings R, i.e. Dedekind domains with only one prime $\mathfrak{p}$. Both maps are well-defined.

"$\leftarrow \circ \rightarrow = \mathrm{id}$" Let Λ as above. Since R is a principal ideal domain, the R-lattice Λ is free. Performing a change of basis, we may assume that $\Lambda = R^n \subset K^n$. We then need to show that $\{x \in K^n \mid x \in \hat{R}_\mathfrak{p}\} = R^n$, which is clear since for $t \in K$, $t \in \hat{R}_\mathfrak{p}$ and $t \in R$ are both equivalent to $v_\mathfrak{p}(t) \geqslant 0$.

"$\rightarrow \circ \leftarrow = \mathrm{id}$" Let $\Lambda_\mathfrak{p} \subset \hat{K}_\mathfrak{p}$ as above, and let $\Lambda = \{x \in K^n \mid x \in \Lambda_\mathfrak{p}\}$. Let $\mathfrak{p}^a \hat{R}_\mathfrak{p}^n \subseteq \Lambda_\mathfrak{p}$. We then have $\mathfrak{p}^a R^n \subseteq \Lambda$, so $\mathfrak{p}^a \hat{R}_\mathfrak{p}^n \subseteq \Lambda \otimes_R \hat{R}_\mathfrak{p}$. Furthermore, for any $x \in \Lambda_\mathfrak{p}$, there exists some $y \in K^n$ such that $x - y \in \mathfrak{p}^a \hat{R}_\mathfrak{p}^n$. Then, we also have $y \in \Lambda_\mathfrak{p}$, so $y \in \Lambda$. But since $\mathfrak{p}^a \hat{R}_\mathfrak{p}^n \subseteq \Lambda \otimes_R \hat{R}_\mathfrak{p}$, it follows that $x \in \Lambda \otimes_R \hat{R}_\mathfrak{p}$. □

Theorem 4 ([Rei03, Corollary 11.6])**.** *Let S be a degree n extension of R. Then, S is a maximal extension of R if and only if $S \otimes_R \hat{R}_\mathfrak{p}$ is a maximal extension of $\hat{R}_\mathfrak{p}$ for all $\mathfrak{p}$.*

Proof. This follows exactly like Theorem 2. □

References

[Rei03] I. Reiner. *Maximal orders.* Vol. 28. London Mathematical Society Monographs. New Series. Corrected reprint of the 1975 original, With a foreword by M. J. Taylor. The Clarendon Press, Oxford University Press, Oxford, 2003, pp. xiv+395. ISBN: 0-19-852673-3.