Math 223a: Algebraic Number Theory

Fall 2020

Problem set #9

due Friday, November 6 at 10:30am

Problem 1. Let $p \geqslant 3$ be any odd prime and let $n \geqslant 4$. Show that $\mathbb{Q}_p$ doesn't have a Galois extension with Galois group S_n. (Feel free to look up the subgroup structure of S_n online.)

Problem 2. Let K be a local field with residue field $\mathbb{F}_q$.

a) Let $n \geqslant 1$. Show that K has a totally and tamely ramified Galois extension with Galois group $\mathbb{Z}/n\mathbb{Z}$ if and only if $n \mid q-1$.

b) Let $n \geqslant 3$. Show that K has a tamely ramified Galois extension whose Galois group is the dihedral group of degree $2n$ if and only if $n \mid q+1$.[1]

Problem 3. Let $n \geqslant 1$ and let K be a local field of characteristic 0. Show that K has only finitely many field extensions of degree n.

Problem 4. Let $K \subseteq \mathbb{Q}(\zeta_\infty)$ be a finite field extension of $\mathbb{Q}$. For any prime p, let $k = k_p$ be the smallest nonnegative integer such that $I^k(\mathfrak{p}|p) = 1$, where $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_K$ dividing p. (In particular, $k_p = 0$ if and only if p is unramified.) Show that the conductor of K is $\prod_p p^{k_p}$.

[1] *Hint:* For "$\Leftarrow$", use an explicit construction. (No need for class field theory!)