Math 223a: Algebraic Number Theory

Fall 2020

Problem set #8

due Friday, October 30 at 10:30am

Problem 1. Let $L|K$ be a finite field extension with Galois group $\mathrm{Gal}(L|K) \cong \mathbb{Z}/n\mathbb{Z}$ generated by σ.

a) Let $a \in L$. Show that there exists some $b \in L$ such that $a = b - \sigma(b)$ if and only if $\mathrm{Tr}_{L|K}(a) = 0$. (*Additive Hilbert 90*)

b) Assume that $\mathrm{char}\, K = n = p$. Show that $L = K(\alpha)$ for a root α of $X^p - X - t$ for some $t \in K$. (*Artin-Schreier theory*)

Problem 2. a) Let $n \geqslant 1$ and let K be any field containing n distinct n-th roots of unity. Let $a \in K^\times$. Show that the field extension $K(\sqrt[n]{a})|K$ has degree n if and only if $a \notin K^{\times p}$ for all prime numbers p dividing n.

b) Let p by a prime number and let K be any field of characteristic $\mathrm{char}\, K \neq p$. Let $a \in K^\times$. Show that the polynomial $X^p - a \in K[X]$ is irreducible if and only if $a \notin K^{\times p}$.

c) Find a number $a \in \mathbb{Q}^\times$ such that the polynomial $X^4 - a \in \mathbb{Q}[X]$ is not irreducible although $a \notin \mathbb{Q}^{\times 2}$.

Problem 3. Denote by $(\cdot, \cdot)_2$ the Hilbert symbol for $K = \mathbb{Q}_2$ and $n = 2$. Show that

$$(2^s \cdot a, 2^t \cdot b) = (-1)^{s \cdot \frac{b^2-1}{8} + t \cdot \frac{a^2-1}{8} + \frac{a-1}{2} \cdot \frac{b-1}{2}}$$

for all $a, b \in \mathbb{Z}_2^\times$ and $s, t \in \mathbb{Z}$.

Problem 4. Show that an element x of $\mathbb{Q}_3^\times$ can be written as

$$x = a^4 - 4a^2b^2 + 4b^4 - 6a^2c^2 - 12b^2c^2 + 9c^4 + 48abcd - 12a^2d^2 - 24b^2d^2 - 36c^2d^2 + 36d^4$$

with $a, b, c, d \in \mathbb{Q}_3$ if and only if $x \in \mathbb{Q}_3^{\times 2}$. (You may use a computer algebra system.)