Math 223a: Algebraic Number Theory

Fall 2020

Problem set #4

due Friday, October 2 at 10:30am

Problem 1. Let $L|K$ be a finite extension of number fields and let M be the Galois closure of L over K. Show that a prime $\mathfrak{p}$ of K splits completely in L if and only if it splits completely in M.

Problem 2. We call an algebraic field extension $L|K$ *abelian* if it is a Galois extension with abelian Galois group.

Let $M|K$ be a Galois extension with Galois group G. Show that $M|K$ has a (unique) *maximal abelian subextension* $T|K$: any subextension $L|K$ of $M|K$ is abelian if and only if $L \subseteq T$.

Show that $\mathrm{Gal}(M|T) = \overline{[G,G]}$ is the topological closure of the commutator subgroup of G.

Problem 3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \dots)$ be the smallest field extension of $\mathbb{Q}$ containing the square roots of all prime numbers.

a) Show that $\mathrm{Gal}(K|\mathbb{Q}) \cong \prod_{k=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, with the product topology (obtained from the discrete topology on $\mathbb{Z}/2\mathbb{Z}$). How does the element of $\mathrm{Gal}(K|\mathbb{Q})$ corresponding to a tuple $(a_k)_k \in \prod_{k=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ act on K?

b) Show that $\mathrm{Gal}(K|\mathbb{Q})$ has a subgroup H of finite index which is not open.

Problem 4. Recall some notation: We say that two sets A and B have the same cardinality (written as $|A| = |B|$) if there is a bijection $A \xrightarrow{\sim} B$. We say that the cardinality of A is at most the cardinality of B (written as $|A| \leqslant |B|$) if there is an injection $A \hookrightarrow B$. This is equivalent to the existence of a surjection $B \twoheadrightarrow A$. We also know that $|A| \leqslant |B|$ and $|B| \leqslant |A|$ implies that $|A| = |B|$. For example, $|\mathbb{N}| < |\mathbb{R}| = |2^{\mathbb{N}}|$ where $2^{\mathbb{N}}$ denotes the set of subsets of $\mathbb{N}$. A set A is countable if and only if $|A| \leqslant |\mathbb{N}|$. (We know that $\overline{\mathbb{Q}}$ is countable because $\overline{\mathbb{Q}} = \bigcup_{f(X) \in \mathbb{Q}[X]} \{\alpha \in \overline{\mathbb{Q}} \mid f(\alpha) = 0\}$ is the union of countably many countable (in fact finite) sets.)

a) Show that $|\,\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})| = |2^{\mathbb{N}}|$.

b) Show that $|\{H \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \text{ open subgroup}\}| = |\mathbb{N}|$.

c) Show that $|\{H \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \text{ closed subgroup}\}| = |2^{\mathbb{N}}|$.

d) (bonus) What is $|\{H \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \text{ subgroup}\}|$?