Math 223a: Algebraic Number Theory

Fall 2020

Problem set #2

due Friday, September 18 at 10:30am

Problem 1. Let K be a nonarchimedean local field with residue field $\kappa_K = \mathbb{F}_q$. Show that K contains $q-1$ distinct $(q-1)$-st roots of unity.

Problem 2. Show that the equation $(x^2+1)(x^2+17)(x^2-17) = 0$ doesn't satisfy the local-global principle over $\mathbb{Z}$: It has a solution in $\mathbb{Z}_p$ for each p and a solution in $\mathbb{R}$, but no solution in $\mathbb{Z}$. (For $p = 2$, wait until the Monday lecture if necessary.)

Problem 3. Let K be a local field with residue field $\mathbb{F}_q$.

a) Equip $\mathbb{Z}$ with the discrete topology. Show that the map $\mathcal{O}_K^\times \times \mathbb{Z} \to K^\times$ sending (x, n) to $x \cdot \pi_K^n$ is a group isomorphism and a homeomorphism.

b) Let $\mu_{q-1} \subset K$ be the group of $(q-1)$-st roots of unity. Show that the map $\mu_{q-1} \times U^{(1)} \to \mathcal{O}_K^\times$ sending (x, y) to xy is a group isomorphism and a homeomorphism.

Problem 4. We call a sequence $(a_0, a_1, \dots)$ *preperiodic* if there exist $n \geqslant 0$ and $k \geqslant 1$ such that $a_i = a_{i+k}$ for all $i \geqslant n$. Write a number $x \in \mathbb{Q}_p$ as $x = p^r \cdot \sum_{i=0}^{\infty} a_i p^i$ with digits $a_0, a_1, \dots \in \{0, \dots, p-1\}$. Show that the digit sequence $(a_0, a_1, \dots)$ is preperiodic if and only if $x \in \mathbb{Q}$.

Problem 5. Let K be a field of characteristic zero which is complete with respect to the discrete valuation v. Assume that the residue field κ_v has characteristic $p \neq 0$.

a) Show that for $x \in K$, the series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges if and only if $v(x) > \frac{v(p)}{p-1}$.

b) Show that for $x \in K$, the series

$$\log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges if and only if $v(x) > 0$.