Math 223a: Algebraic Number Theory

Fall 2020

Some ideas for final papers

Here are some ideas for the 7–10 page final papers. You are of course more than welcome to come up with your own topics!

1. *Witt vectors:* They provide a way to construct the ring $\mathbb{Z}_p$ "directly" from its residue field $\mathbb{F}_p$, and its unramified degree n extension $\mathbb{Z}_{p^n}$ from the residue field $\mathbb{F}_{p^n}$. Witt managed to use this to explicitly describe cyclic extensions of K of degree p^n when $\mathrm{char}(K) = p$. (The case $n = 1$ is called Artin–Schreier theory.) See for example [Bos18, Sections 4.8–4.10].

2. *Complex multiplication:* We have very explicitly constructed the maximal abelian extension K^{ab} of a field K when K is a local field or $K = \mathbb{Q}$. The case of general number fields is more difficult. The theory of complex multiplication provides a way to construct K^{ab} when K is a quadratic imaginary number fields. This involves looking at particular elliptic curves associated to K. (*Kronecker's Jugendtraum*) See for example [Cox13] or [67, Chapter XIII] or [ST15, Chapter 6] or [Sil94, Chapter II].

3. *Tropical geometry:* This can be viewed as a generalization of Newton polygons. Say we have polynomials $f_1, \ldots, f_k \in \mathcal{O}_K[X_1, \ldots, X_n]$. Can we determine the set of possible valuation tuples $(v_K(\alpha_1), \ldots, v_K(\alpha_n)) \in \mathbb{R}^n$ for solutions $\alpha = (\alpha_1, \ldots, \alpha_n) \in \overline{K}^n$ to $f_1(\alpha) = \cdots = f_k(\alpha) = 0$? (Newton polygons completely answered that question for $k = n = 1$.) If you're interested in some general geometry of tropical varieties, look for example at [RST05] or [Mik04]. If you're interested in applications to nonarchimedean local fields, look for example at the fundamental theorem of tropical geometry and the transverse intersection theorem in [MS15, Chapter 3].

4. *Cubic and higher reciprocity laws:* This is a generalization of the quadratic reciprocity law that can be tackled using Hilbert symbols coming from the Artin reciprocity map. See for example [Cox13] or [Lem00].

5. *Hasse–Minkowski theorem:* Show the Hasse principle for varieties of the form $\{f(X_1, \ldots, X_n) = 0\}$, where $f(X_1, \ldots, X_n)$ is a homogeneous polynomial of degree two. See for example [Ser73, Part I]. (There are many other very interesting instances of the Hasse principle! For example, if you like analytic number theory, I can suggest topics using the circle method.)

6. *Some more local analysis:* You can define a so-called Haar measure on nonarchimedean local fields and prove theorems very similar to familiar theorems over the real numbers, such as a change of variables formula for integration. See for example [Igu00] or [Rob00].

References

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