8.9. Homology

Thm/Def There is a unique family of homology functors

$$H_i(G,\cdot) : \{G\text{-mod.}\} \longrightarrow \{\text{ab. grp.}\} \quad (i \geq 1)$$

satisfying the following axioms:

a) If $0 \to A \to B \to C \to 0$ is a short exc. seq. of G-modules, we obtain a long exc. seq.

$$\begin{array}{l} \cdots \to H_2(G,B) \to H_2(G,C) \xrightarrow{\delta} \\ \to H_1(G,A) \to H_1(G,B) \to H_1(G,C) \xrightarrow{\delta} \\ \to A_G \to B_G \to C_G \to 0 \end{array}$$

b) If A is an induced G-module, then $H_i(G,A)=1$ $\forall i \geq 1$.

c) A comm. diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

of short ex. seq. induces a comm. diagram

$$\begin{array}{cccccccccc} \cdots & \to H_1(G,C) & \xrightarrow{\delta} & A_G & \to & B_G & \to & C_G & \to & 0 \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to H_1(G,C') & \xrightarrow[\delta]{} & A'_G & \to & B'_G & \to & C'_G & \to & 0 \end{array}$$

By convention, we set $H_0(G,A) = A_G$.

Pf As for cohomology, but replace $\mathrm{Hom}_G(-,A)$ by $- \otimes_{\mathbb{Z}[G]} A$ and reverse some arrows. $\square$

Rmk. One can again define chains/cycles/boundaries using the standard resolution. H_i = cycles/boundaries.

Def The kernel of the augmentation map (a ring hom.)

$$\varepsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z}$$
$$\sum_{g\in G} \underbrace{a_g}_{\in\mathbb{Z}} g \longmapsto \sum_{g\in G} a_g$$

is called the augmentation ideal I_G.

Rmk. I_G has $\mathbb{Z}$-basis $(g-e)_{e\neq g\in G}$.

Pf $\sum_{g\in G} a_g g = \sum_{g\in G} a_g (g-e)$. $\square$

Cor $A_G = A / \langle \underbrace{ga-a}_{(g-e)\cdot a} \mid g\in G, a\in A\rangle = A / I_G\cdot A$

Lemma $H_1(G,\mathbb{Z}) \cong I_G / I_G\cdot I_G$

Pf $0 \to I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

$x \longmapsto x$

$\mathbb{Z}[G]$: induced G-mod.

$\cdots \to \underbrace{H_1(G,\mathbb{Z}[G])}_{=0} \longrightarrow H_1(G,\mathbb{Z}) \xrightarrow{\delta}$

$$\begin{array}{ccccc}
(I_G)_G & \longrightarrow & \mathbb{Z}[G]_G & \longrightarrow & \mathbb{Z}_G \\
\| & & \| & & \| \\
H_0(G,I_G) & & H_0(G,\mathbb{Z}[G]) & & H_0(G,\mathbb{Z}) \\
\| & & \| & & \| \\
I_G/I_G\cdot I_G & & \mathbb{Z}[G]/I_G\cdot\mathbb{Z}[G] & & \mathbb{Z} \\
 & & \| & & \\
x & \longmapsto & x = 0 \quad \mathbb{Z}[G]/I_G & &
\end{array}$$

$\Rightarrow H_1(G,\mathbb{Z}) = \ker(I_G/I_G^2 \to \mathbb{Z}[G]/I_G)$ $\square$

Lemma $I_G/I_G\cdot I_G \cong \mathbb{Z}(G)/\langle g_1g_2-g_1-g_2 \mid g_1,g_2\in G\rangle_{\mathbb{Z}\text{-mod}} \cong G^{ab}$

$$[g-e] \longleftrightarrow \underset{\text{additive}}{[g]} \longleftrightarrow \underset{\text{multiplicative}}{[g]}$$

Pf $\underbrace{(g_1-e)(g_2-e)}_{\text{generate } I_G\cdot I_G} = (g_1g_2-e)-(g_1-e)-(g_2-e)$

$\mathbb{Z}(G)/\langle g_1g_2-g_1-g_2 \mid g_1,g_2\in G\rangle$ is the max. ab. quotient of G. $\square$

8.10. Transfer map

Lemma C Let $H\leq G$ be finite groups. Recall the transfer map $V: G^{ab} \longrightarrow H^{ab}$

$t \longmapsto \prod_{i=1}^{n} [h_i]$, where $r_1,\dots,r_n\in G$ are repr. of the cosets in $H\backslash G$ and $r_i t = h_i r_{\pi(i)}$, $h_i\in H$, $\pi\in S_n$.

The following diagram commutes:

$$\begin{tikzcd}
G^{ab} \arrow[r, "V"] \arrow[dd, "\wr"'] & H^{ab} \arrow[d, "\wr"] \\
 & I_H/I_H\cdot I_H \arrow[d, hook, "\Leftarrow \text{injective!}"] \\
I_G/I_G\cdot I_G \arrow[r] & I_G/I_H\cdot I_G
\end{tikzcd}$$

$$x \longmapsto \sum_{i=1}^{n} r_i\cdot x$$

Pf $t-e \longmapsto \sum_i r_i(t-e) = \sum(r_it-r_i) = \sum(h_ir_{\pi(i)}-r_i)$

$\equiv \sum(h_i+\cancel{r_{\pi(i)}}-\cancel{r_i}-e) = \sum(h_i-e)$

mod $I_H\cdot I_G$ $\square$

Remark

$$\begin{array}{ccc} G^{ab} & \xrightarrow{V} & H^{ab} \\ \| \wr & & \wr\| \\ H_1(G,\mathbb{Z}) & \xrightarrow[\text{Res}]{} & H_1(H,\mathbb{Z}) \end{array}$$

Res : $H_i(G, A) \longrightarrow H_1(H, A)$ is the composition

$$H_i(G, A) \longrightarrow H_i(G, \text{coind}_H^G A) \xrightarrow{\sim} H_i(H, A)$$

arising from $A \to \text{coind}_H^G A$ co-Shapiro

For the principal ideal theorem, we used:

Thm Let G be a finite group and $H = [G, G]$ its commutator subgroup. Then $V : G^{ab} \to H^{ab}$ is the "zero map": $V([g]) = [e]$ $\forall [g] \in G^{ab}$.

Reference: Witt, Verlagerung von Gruppen und Hauptidealsatz.

Pf $G/H = G^{ab} = I_G / I_G \cdot I_G$.

Recall: $I_G = \bigoplus_{e \neq g \in G} \mathbb{Z} \cdot (g - e) \cong \mathbb{Z}^{|G|-1}$.

$\Rightarrow I_G \cdot I_G \subseteq I_G$ is a subgroup of index $[G:H]$.

$\Rightarrow I_G \cdot I_G \cong \mathbb{Z}^{|G|-1}$ as groups.

Let $\left(\underbrace{\sum_{e \neq g \in G} \underbrace{m_{g'g}}_{\in \mathbb{Z}} (g - e)}_{I_G} \right)_{e \neq g' \in G}$ be a $\mathbb{Z}$-basis of $I_G \cdot I_G$.

$\Rightarrow \det(m_{g'g})_{e \neq g', g \in G} = \pm [G:H]$.

w.l.o.g. +

Since $\sum_g m_{g'g} \underbrace{(g-e)}_{\text{basis of } I_G} \in I_G \cdot I_G$, we can

find $\mu_{g'g} \in \mathbb{Z}[G]$ s.t. $\mu_{g'g} \equiv m_{g'g} \mod I_G$ (1)

and $\sum_g \mu_{g'g}(g-e) = 0 \quad \forall_{e \neq} g' \in G$. (2)

Idea: solve the system of lin. eq. (2) for $g-e$.

Problem: $\mathbb{Z}[G]$ isn't commutative!

But $\mathbb{Z}[G/H] \cong \mathbb{Z}[G^{ab}]$ is commutative.

Claim We have a ring isomorphism

$$\mathbb{Z}[G] / \underbrace{I_H \cdot G}_{\text{two-sided } \mathbb{Z}[G]\text{-ideal generated by } I_H} \xrightarrow[\varphi]{\sim} \mathbb{Z}[G/H]$$

because H is a normal subgroup of G

Pf

$$\left[\sum_g a_g g\right] \longmapsto \sum_g a_g \overbrace{(Hg)}^{\in G/H}$$

$\sum a_g (Hg) = 0$ in $\mathbb{Z}[G/H]$

$\quad \shortparallel$

$\sum_i \sum_h a_{hr_i} (Hr_i)$

$\Leftrightarrow \sum_h a_{hr_i} = 0 \ \forall i \Rightarrow \sum a_g g = \sum a_{hr_i} h r_i$

$$= \underbrace{\sum_i \underbrace{\left(\sum_h a_{hr_i} h\right)}_{\in I_H} \underbrace{r_i}_{\in G}}_{\in I_H \cdot G}.$$

$\square$

...

$\rightsquigarrow$ We can interpret $(\mu_{g'g})_{g',g}$ as a matrix with entries in $\mathbb{Z}[G/H]$ and construct its adjoint matrix $(\lambda_{g'g})_{g',g}$ with entries in $\mathbb{Z}[G/H]$.

Lift them to entries in $\mathbb{Z}[G]$ using ρ.

The product of the matrices over $\mathbb{Z}[G/H]$ is $\det(\mu_{g'g})_{g'g}$ times the identity matrix.

$$\Rightarrow \sum_{g'} \lambda_{g''g'} \mu_{g'g} \equiv \begin{cases} \det(\mu_{g'g})_{g',g}, & g''=g \\ 0, & g'' \neq g \end{cases} \mod I_H \cdot G.$$

$$\Rightarrow \sum_{g',g} \lambda_{g''g'} \mu_{g'g} \underbrace{(g-e)}_{\in I_G} \overset{(2)}{=\!=} 0$$

$$\equiv \underbrace{\det(\mu_{g'g})_{g',g}}\cdot \underbrace{(g''-e)}_{\in I_G} \mod I_H \cdot G \cdot I_G$$

$$I_G \subseteq I_H \cdot G$$

$$\underset{(1)}{\equiv} \det(m_{g'g})_{g',g} \mod I_H \cdot G$$

$$\equiv [G:H] \quad \mod I_H \cdot G$$

$$\equiv \sum_{i=1}^{n} r_i \quad \mod I_G \cdot I_H \cdot G$$

$$\Rightarrow \sum_i r_i (g''-e) \in I_H \cdot G \cdot I_G = I_H \cdot I_G.$$

$$\Rightarrow V([g'']) = 0 = (e)$$

$\square$

Lemma C