Thm (Frobenius reciprocity)

For $H \leq G$ of finite index and any G-module A and H-module B,

$$\mathrm{Hom}_G(A, \mathrm{Ind}_H^G B) \cong \mathrm{Hom}_H(A, B).$$

only depends on action of H (not G) on A!

Pf $\quad a \mapsto \underbrace{(g \mapsto \phi(a)(g))}_{\phi(a)} \longmapsto (a \mapsto \phi(a)(e))$

$a \mapsto (g \mapsto f(ga)) \longleftarrow f$ $\square$

Rmk The functor $\mathrm{Ind}_H^G : \{H\text{-mod.}\} \longrightarrow \{G\text{-mod}\}$ is exact (sends short exact seq. of H-mod. to short exact seq. of G-mod.).

Thm (Shapiro's lemma)

Let $H \leq G$ of finite index and let A be an H-mod. Then, there is a (canonical) isomorphism

$$H^n(G, \mathrm{Ind}_H^G A) \cong H^n(H, A).$$

Pf Let $0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots$ be a resolution by free G-modules. It's also a resolution by free H-modules (because $\mathbb{Z}[G] = \bigoplus_{g \in G/H} g\mathbb{Z}[H]$ is a free $\mathbb{Z}[H]$-module).

$$\mathrm{Hom}_G(P_i, \mathrm{Ind}_H^G A) \cong \mathrm{Hom}_H(P_i, A)$$

Frobenius reciprocity

These isom. commute with the differential maps d^i.

(as constructed above) $\square$

Ex ($n=0$) $\quad (\mathrm{Ind}_H^G A)^G \cong A^H.$

Def For $H \leq G$ of finite index and any G-module A,

the hom. $\operatorname{Ind}_H^G A \longrightarrow A$ (depends only on H-action!) of G-mod.

$$g \otimes a \longmapsto ga$$

induces a hom. $H^n(G, \operatorname{Ind}_H^G A) \longrightarrow H^n(G, A)$ of groups.

Then, the corestriction map is the composition

$$\mathrm{cor}: H^n(H, A) \cong H^n(G, \operatorname{Ind}_H^G A) \longrightarrow H^n(G, A).$$

Ex $\mathrm{cor}: H^0(H, A) \longrightarrow H^0(G, A)$

$$\begin{array}{ccc} H^0(H,A) & \longrightarrow & H^0(G,A) \\ \| & & \| \\ A^H & & A^G \\ a & \longmapsto & \displaystyle\sum_{g \in G/H} ga \end{array}$$

Thm $\mathrm{cor} \circ \mathrm{Res}$ is the mult. by $[G:H]$ map.

$$H^n(G, A) \underset{\mathrm{cor}}{\overset{\mathrm{Res}}{\rightleftarrows}} H^n(H, A).$$

Pf $\mathrm{cor} \circ \mathrm{Res}$ is induced by

$$\operatorname{Hom}_G(P_i, A) \xrightarrow{\mathrm{Res}} \operatorname{Hom}_H(P_i, A) \cong \operatorname{Hom}_G(P_i, \operatorname{Ind}_H^G A) \longrightarrow \operatorname{Hom}_G(P_i, A)$$

$$f \longmapsto f \longmapsto \Big(p \mapsto \underbrace{(g \mapsto f(gp))}_{= \sum_{g \in H\backslash G} g^{-1} \otimes f(gp)}\Big) \longmapsto \underbrace{\underbrace{\Big(p \mapsto \sum_{g \in H\backslash G} \underbrace{g^{-1} f(gp)}_{f(p)}\Big)}_{[G:H] \cdot f(p)}}_{[G:H] \cdot f}$$

because f is G-mod. hom.

$\square$

Rmk $H^n(1, A) = \begin{cases} A & , n=0 \\ 0 & , n\geq 1 \end{cases}$ (because A is a coinduced 1-module)

any abelian group

Cor If $|G|<\infty$, then $|G|\cdot H^n(G,A)=0 \;\forall n\geq 1$.

Pf Apply the lemma with $H=1$:

$$H^n(G,A) \underset{\text{Cor}}{\overset{\text{Res}}{\rightleftarrows}} H^n(1,A)=0$$ □

Cor If the mult. by $|G|$ map $A \to A$ is an isomorphism (e.g. $A=\mathbb{Q}$ or fin. ab. group of order coprime to $|G|$), then $H^n(G,A)=0 \;\forall n\geq 1$.

Pf $A \xrightarrow{|G|\cdot} A$ isom.

$\Rightarrow H^n(G,A) \xrightarrow{|G|\cdot} H^n(G,A)$ isom. and zero (by prev. Cor.) □

Cor If $|G|<\infty$, then $H^n(G,\mathbb{Q}/\mathbb{Z}) \cong H^{n+1}(G,\mathbb{Z}) \;\forall n\geq 1$.

Pf $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

$$\cdots \to \underset{=0}{H^n(G,\mathbb{Q})} \to H^n(G,\mathbb{Q}/\mathbb{Z}) \to H^{n+1}(G,\mathbb{Z}) \to \underset{=0}{H^{n+1}(G,\mathbb{Q})} \to \cdots$$ □

Thm Let H be a normal subgroup of G and let A be a G-module. Let $n \geq 1$ such that $H^i(H,A)=0$ for $i=1,\dots,n-1$. Then, we obtain the inflation-restriction exact sequence

$$0 \to H^n(G/H, A^H) \xrightarrow{\text{Inf}} H^n(G,A) \xrightarrow{\text{Res}} H^n(H,A).$$

Pf Induction over n:

$n=1$:

$$\begin{array}{c} 0 \\ \downarrow \\ H^1(G/H, A^H) = \{ (\underset{\overset{\cap}{A^H}}{a_g})_{g\in G/H} \mid a_{g_1g_2} = a_{g_1} + g_1 a_{g_2} \} / \{ (gb-b)_{g\in G/H} \mid b \in A^H \} \\ \downarrow \text{Inf} \\ H^1(G,A) = \{ (\underset{\overset{\cap}{A}}{a_g})_{g\in G} \mid \dots \} / \{ (gb-b)_{g\in G} \mid b\in A \} \\ \downarrow \text{Res} \\ H^1(H,A) = \{ (\underset{\overset{\cap}{A}}{a_g})_{g\in H} \mid \dots \} / \{ (gb-b)_{g\in H} \mid b \in A \} \end{array}$$

Inf is injective: Let $(a_g)_{g\in G/H} \in \ker(\text{Inf})$.

$\Rightarrow \exists b \in A : \forall g \in G : a_{gH} = gb-b$

$\Downarrow$

$\forall g \in H : \underset{\overset{\|}{0}}{a_H} = gb-b \Rightarrow b \in A^H$

$\Rightarrow (a_g) = 0$ in $H^1(G/H, A^H)$.

...

$n-1 \longrightarrow n$: Use construction 1 of cohom.

Let $A^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) = \{\text{maps } G \to A\}$
(coinduced),

$A \hookrightarrow A^*$ G-mod. hom. as before.

$a \longmapsto (g \mapsto ga)$

$$0 \to A \longrightarrow A^* \longrightarrow A^*/A \to 0$$

$$1 = H^k(G, A^*) \to H^k(G, A^*/A) \to H^{k+1}(G, A) \to H^{k+1}(G, A^*) = 1 \qquad \forall k \geq 1$$

$\Rightarrow H^k(G, A^*/A) \cong H^{k+1}(G, A) \quad \forall k \geq 1.$

(Same with G replaced by H ...)

$\Rightarrow H^i(H, A^*/A) = 0$ for $i = 1, \dots, n-2$ by assumption.

By induction,

$$\begin{array}{ccccccc}
0 \to & H^{n-1}(G/H, (A^*/A)^H) & \xrightarrow{\mathrm{Inf}} & H^{n-1}(G, A^*/A) & \xrightarrow{\mathrm{Res}} & H^{n-1}(H, A^*/A) \\
 & \wr\wr & & \wr\wr & & \wr\wr \\
0 \to & H^n(G/H, A^H) & \xrightarrow{\mathrm{Inf}} & H^n(G, A) & \xrightarrow{\mathrm{Res}} & H^n(H, A)
\end{array}$$

$\Rightarrow$ bottom row exact. $\square$

8.8. Cup products

Def Let M, N be G-modules and $r, s \geq 0$. Then, $M \otimes_{\mathbb{Z}} N$ is also a G-module ($g(m \otimes n) = (gm) \otimes (gn)$).

Define the <u>cup product</u>

$$\cup: H^r(G, M) \times H^s(G, N) \longrightarrow H^{r+s}(G, M \otimes_{\mathbb{Z}} N)$$

by letting

$$(f_1 \cup f_2)(g_0, \dots, g_{r+s}) = \underbrace{f_1(g_0, \dots, g_r)}_{\in M} \otimes \underbrace{f_2(g_r, \dots, g_{r+s})}_{\in N}$$

for homogeneous cocycle $f_1 \in \tilde{C}^r(G, M)$, $f_2 \in \tilde{C}^s(G, N)$

$$\left(\rightsquigarrow f_1 \cup f_2 \in \tilde{C}^{r+s}(G, M \otimes_{\mathbb{Z}} N) \right).$$

Ex $(r = s = 0)$

$$\cup: M^G \times N^G \longrightarrow (M \otimes_{\mathbb{Z}} N)^G$$

$$(m, n) \longmapsto m \otimes n$$

Rmk $(x \cup y) \cup z = x \cup (y \cup z)$

$x \cup y = (-1)^{rs} \, y \cup x$ for $x \in H^r(G, M)$, $y \in H^s(G, N)$

(identifying $M \otimes N = N \otimes M$).

For $H \leq G$: $\mathrm{Res}_H(x \cup y) = \mathrm{Res}_H(x) \cup \mathrm{Res}_H(y)$

$\mathrm{Cor}(x \cup \mathrm{Res}(y)) = \mathrm{Cor}(x) \cup y$.

(Try this out with $M = \mathbb{Z}$, $r = 0$ ...)