Def A <u>free G-module</u> is a free $\mathbb{Z}[G]$-module, i.e. $\bigoplus_{i \in I} \mathbb{Z}[G]$ for any set I.

A <u>coinduced G-module</u> is a module of the form

$$\mathrm{Hom}_{\text{group}}(\underset{G}{\overset{\circlearrowleft}{\mathbb{Z}[G]}}, X) \text{ for some abelian group } X.$$

X ← gives group structure

$$= \{\text{map } G \to X\} = \{(x_g)_{g \in G} \mid x_g \in X \ \forall g\}$$

(not necessarily hom.)

$$= \left\{ \sum_{g \in G} x_g g \mid x_g \in X \ \forall g \right\}$$

$$\left(\text{action given by } h \sum_g x_g g = \sum_g x_g hg = \sum_g x_{h^{-1}g}\, g \right)$$

An <u>induced G-module</u> is a module of the form

$$\underset{G}{\overset{\circlearrowleft}{\mathbb{Z}[G]}} \otimes_{\mathbb{Z}} X \text{ for some abelian group } X.$$

$$= \{(x_g)_{g \in G} \mid x_g \in X \ \forall g, \ x_g = 0 \text{ for all but fin. many } g\}.$$

(same action as before)

<u>Rmk</u> For finite groups G, induced = coinduced.

<u>Ex</u> $(\mathbb{Z}/2\mathbb{Z})[G]$ is an induced G-module, but not free.

8.3. Cohomology

[Reference: Milne's notes of CFT, Neukirch's books on CFT, ...]

<u>Thm/Def</u> There is a unique family of <u>cohomology functors</u> $H^i(G,\cdot): \{G\text{-mod.}\} \longrightarrow \{\text{ab. grp.}\}$ $(i \geq 1)$

satisfying the following axioms:

a) If $0 \to A \to B \to C \to 0$ is an ex. seq. of G-mod., we obtain a long ex. seq.

$$0 \to A^G \longrightarrow B^G \longrightarrow C^G \to H^1(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C) \to H^2(G,A) \longrightarrow H^2(G,B) \longrightarrow H^2(G,C) \to \cdots$$

b) If A is coinduced, then $H^i(G,A)=0 \quad \forall i \geq 1$.

c) Any comm. diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

of short ex. seq. produces a comm. diagram of long ex. seq.

$$\begin{tikzcd} 0 \arrow[r] & A^G \arrow[r] \arrow[d] & B^G \arrow[r] \arrow[d] & C^G \arrow[r] \arrow[d] & H^1(G,A) \arrow[r] \arrow[d] & H^1(G,B) \arrow[r] \arrow[d] & \cdots \arrow[d] \\ 0 \arrow[r] & A'^G \arrow[r] & B'^G \arrow[r] & C'^G \arrow[r] & H^1(G,A') \arrow[r] & H^1(G,B') \arrow[r] & \cdots \end{tikzcd}$$

By convention, we set $H^0(G,A) = A^G$.

Sketch of pf

Uniqueness / construction 1

Consider the injective hom. of G-modules

$$A \hookrightarrow \{\text{map } G \to A\} = A^* .$$
$$a \longmapsto (g \mapsto g^{-1}a)$$

We're ignoring the action of G on A, here!

It is a G-module hom.:

$$ha \longmapsto (g \mapsto g^{-1}ha)$$
$$= (hg \mapsto g^{-1}a)$$
$$= h\cdot(g \mapsto g^{-1}a).$$

The short exc. seq.

$$0 \to A \longrightarrow A^* \longrightarrow A^*/A \longrightarrow 0$$

gives rise to

$$0 \to A^G \longrightarrow (A^*)^G \longrightarrow (A^*/A)^G \to$$
$$\to H^1(G,A) \to \underset{\substack{\| (b) \\ 0}}{H^1(G,A^*)} \to H^1(G,A^*/A) \to$$
$$\to H^2(G,A) \to \underset{\substack{\| (b) \\ 0}}{H^2(G,A^*)} \to H^2(G,A^*/A) \to$$
$$\to \cdots$$

$\Rightarrow H^1(G,A) \cong \operatorname{coker}\left((A^*)^G \to (A^*/A)^G\right)$

$\Rightarrow H^1(G,A)$ uniquely determined $\forall A$.

$\Rightarrow H^2(G,A) \cong H^1(G,A^*/A)$

$\Rightarrow H^2(G,A)$ uniquely determined $\forall A$

$\vdots$

Axiom c) shows uniqueness for morphisms $H^i(G,A) \to H^i(G,B)$.

Construction 2

Choose a resolution of $\mathbb{Z}$ by free G-modules:

An ex. sequence

$$0 \leftarrow \mathbb{Z} \xleftarrow{d^0} P_0 \xleftarrow{d^1} P_1 \xleftarrow{d^2} P_2 \xleftarrow{d^3} \cdots$$

where each P_i is a free G-module.

This produces a cochain complex (composition of two consecutive maps is 0)

$$0 \xrightarrow{d^0} \mathrm{Hom}_G(P_0, A) \xrightarrow{d^1} \mathrm{Hom}_G(P_1, A) \xrightarrow{d^2} \mathrm{Hom}_G(P_2, A) \xrightarrow{d^3} \cdots$$

$$\begin{tikzcd} P_i \arrow[d, "f"'] & P_{i+1} \arrow[l, "d^{i+1}"'] \arrow[dl, dashed, "d^{i+1}(f)"] \\ A & \end{tikzcd}$$

It might not be exact, though!

Let $H^i(G, A) = \ker(d^{i+1}) / \mathrm{im}(d^i)$.

Note: $H^0(G, A) = \ker(d^1) = \{ \underset{G\text{-mod. hom.}}{f : P_0 \to A} \mid f \circ d^1 = 0 \}$.

$$= \mathrm{Hom}_G(P_0 / d^1(P_1), A)$$

$$= \mathrm{Hom}_G(\mathbb{Z}, A) = A^G.$$

$$f \longmapsto f(1)$$

$$(n \mapsto nx) \longleftarrow x$$

Now, check the axioms:

b) Let A be coinduced:

$A = \{\text{map } G \to X\}$ for some ab. grp. X.

$$\mathrm{Hom}_G(P_i, A) = \mathrm{Hom}_{\text{group}}(P_i, X)$$

$$(p \mapsto a(p)) \longmapsto (p \mapsto a(p)(e))$$

$$(p \mapsto (g \mapsto x(g^{-1}p))) \longleftarrow\!\shortmid (p \mapsto x(p))$$

$$\begin{tikzcd}
\cdots & P_{i-1} \arrow[l] \arrow[dr, dashed, "g"'] & P_i \arrow[l, "d^i"'] \arrow[d, "f"] & P_{i+1} \arrow[l, "d^{i+1}"'] \arrow[dl, "0"] \\
 & & X &
\end{tikzcd}$$

el. of $\ker(d^{i+1} : \mathrm{Hom}_G(P_i, A) \to \mathrm{Hom}_G(P_{i+1}, A))$

Each P_i is a free $\mathbb{Z}[G]$-module and therefore a free $\mathbb{Z}$-module. $\Rightarrow \exists g$ s.t. $f = g \circ d^i$

$\Rightarrow f \in \mathrm{im}(d^i)$.

$\Rightarrow$ The cochain complex is exact.

$\Rightarrow H^i(G, A) = 0 \quad \forall i \geq 1$.

a) P_i free G-module: $P_i \cong \bigoplus_{i \in I} \mathbb{Z}(G)$

$\Rightarrow \mathcal{H}om_G(P_i, A) \cong \prod_{i \in I} A$

$\Rightarrow$ If $0 \to A \to B \to C \to 0$ is an ex. seq. of G-mod.,

then
$$\begin{array}{ccccccc} 0 \to & \mathcal{H}om_G(P_i, A) & \to & \mathcal{H}om_G(P_i, B) & \to & \mathcal{H}om_G(P_i, C) & \to 0 \\ & \wr\wr & & \wr\wr & & \wr\wr & \\ & \prod_{i \in I} A & & \prod_{i \in I} B & & \prod_{i \in I} C & \end{array}$$

is also an exact sequence.

Apply the snake lemma to

$$\begin{array}{ccccccc} 0 \to & \mathcal{H}om_G(P_i, A) & \to & \mathcal{H}om_G(P_i, B) & \to & \mathcal{H}om_G(P_i, C) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{H}om_G(P_{i+1}, A) & \to & \mathcal{H}om_G(P_{i+1}, B) & \to & \mathcal{H}om_G(P_{i+1}, C) & \to 0 \end{array}$$

This produces the long exact sequence . . .

"$\square$"

8.4. Standard resolution

Here's a resolution of $\mathbb{Z}$ by free G-modules:

$$0 \leftarrow \mathbb{Z} \xleftarrow{d^0} \underset{\substack{\| \\ P_0}}{\mathbb{Z}[G]} \xleftarrow{d^1} \underset{\substack{\| \\ P_1}}{\mathbb{Z}[G^2]} \xleftarrow{d^2} \underset{\substack{\| \\ P_2}}{\mathbb{Z}[G^3]} \leftarrow \cdots$$

Note: $P_i = \mathbb{Z}[G^{i+1}] = \left\{ \sum_{g_0,\dots,g_i \in G} \underbrace{a_{g_0,\dots,g_i}}_{\in \mathbb{Z}} (g_0,\dots,g_i) \right\}$

with G-action $g(g_0,\dots,g_i) = (gg_0,\dots,gg_i)$ is a free $\mathbb{Z}(G)$-module with $\mathbb{Z}[G]$-module basis
$\{(1,g_1,\dots,g_i) \mid g_1,\dots,g_i \in G\}$.

Rmk: The P_i "correspond to" standard simplices in the definition of singular cohomology.

Let $d^i : P_i \longrightarrow P_{i-1}$

$$(g_0,\dots,g_i) \longmapsto \sum_{j=0}^{i} (-1)^j \underset{\substack{\| \\ (g_0,\dots,g_{j-1},g_{j+1},\dots,g_i)}}{(g_0,\dots,\widehat{g_j},\dots,g_i)}$$

It's easy to show $d^i \circ d^{i+1} = 0$ (so $\ker(d^i) \supseteq \operatorname{im}(d^{i+1})$).
To show $\ker(d^i) \subseteq \operatorname{im}(d^{i+1})$, use the chain homotopy maps $h^i : P_i \longrightarrow P_{i+1}$, $(g_0,\dots,g_i) \longmapsto (1,g_0,\dots,g_i)$, which satisfy $d^{i+1} \circ h^i + h^{i-1} \circ d^i = \mathrm{id}$.

If $a \in \ker(d^i)$, then

$$a = d^{i+1}(h^i(a)) + h^{i-1}(\underbrace{d^i(a)}_{0}) = d^{i+1}(h^i(a)) \in \mathrm{im}(d^{i+1}).$$

G-mod. hom. condition

$$\tilde{C}^i(G,A) := \mathrm{Hom}_G(P_i, A) = \{ \underbrace{\tilde{f} : G^{i+1} \to A}_{\text{map}} \mid \tilde{f}(gg_0,\dots,gg_i) = g\tilde{f}(g_0,\dots,g_i) \ \forall g, g_0,\dots,g_i \in G \}$$

<u>(group of homogeneous i-cochains)</u>

$d^i : \tilde{C}^{i-1}(G,A) \to \tilde{C}^i(G,A)$ is given by

$$(d^i \tilde{f})(g_0,\dots,g_i) = \sum_{j=0}^{i} (-1)^j \tilde{f}(g_0,\dots,\hat{g_j},\dots,g_i).$$

$$\begin{array}{l} \tilde{C}^i(G,A) \\ \quad \cup\!\mid \\ \tilde{Z}^i(G,A) = \ker(d^{i+1}) \quad \text{(group of hom. i-cocycles)} \\ \quad \cup\!\mid \\ \tilde{B}^i(G,A) = \mathrm{im}(d^i) \quad \text{(group of hom. i-coboundaries)} \end{array}$$

$$H^i(G,A) = \tilde{Z}^i(G,A) / \tilde{B}^i(G,A).$$

In practice, inhomogeneous cochains tend to be more convenient:

$$C^i(G,A) := \{ (\underbrace{a_{g_1,\dots,g_i}}_{\in A})_{g_1,\dots,g_i \in G} \}$$

There's a group isomorphism

$$\tilde{C}^i(G,A) \cong C^i(G,A)$$

$$\tilde{f} \longleftrightarrow a$$

given by $a_{g_1,\dots,g_i} = \tilde{f}(1, g_1, g_1 g_2, \dots, g_1 \cdots g_i)$.

$$\begin{tikzcd}
0 \arrow[r] & \tilde{C}^0(G,A) \arrow[r, "d^1"] \arrow[d, "\wr"] & \tilde{C}^1(G,A) \arrow[r, "d^2"] \arrow[d, "\wr"] & \tilde{C}^2(G,A) \arrow[r] \arrow[d, "\wr"] & \dots \\
0 \arrow[r] & C^0(G,A) \arrow[r] & C^1(G,A) \arrow[r] & C^2(G,A) \arrow[r] & \dots
\end{tikzcd}$$

$$C^0(G,A) \cong A$$

$$d^1 : \overset{C^0(G,A)}{\overset{\shortparallel}{A}} \longrightarrow C^1(G,A)$$

$$a \longmapsto (ga - a)_{g \in G}$$

$$d^2 : C^1(G,A) \to C^2(G,A)$$

$$(a_g)_{g\in G} \mapsto (a_g + g a_h - a_{gh})_{g,h \in G}$$

$$d^3 : C^2(G,A) \to C^3(G,A)$$

$$(a_{g,h})_{g,h \in H} \mapsto (g a_{h,i} - a_{gh,i} + a_{g,hi} - a_{g,h})_{g,h,i \in G}$$

$$\vdots$$

$$C^i(G,A)$$

$$\cup\!|$$

$$Z^i(G,A) = \ker(d^{i+1}) \quad (\underline{\text{group of inhom. } i\text{-cocycles}})$$

$$\cup\!|$$

$$B^i(G,A) = \operatorname{im}(d^i) \quad (\underline{\text{group of inhom. } i\text{-coboundaries}})$$

$$H^i(G,A) = Z^i(G,A)/B^i(G,A)$$

<u>Ex</u>

$$\left.\begin{array}{l} Z^0(G,A) = \{a \in A \mid \underset{\substack{\Updownarrow \\ ga = a}}{ga - a = 0} \ \forall g \in G\} = A^G \\[2em] B^0(G,A) = 0 \end{array}\right\} \Rightarrow H^0(G,A) = A^G.$$

$$\left.\begin{array}{l} Z^1(G,A) = \{(a_g)_{g \in G} \mid a_{gh} = a_g + g a_h \ \forall g,h\} \\ B^1(G,A) = \{(ga - a)_{g \in G} \mid a \in A\} \end{array}\right\} \text{as before.}$$