8. Group (co-)homology

8.1. G-modules

Def Let G be a finite group (written multiplicatively). A (left) <u>G-module</u> is an abelian group A (additively) with a left action of G on A s.t. $g(a+a') = ga + ga'$ $\forall g \in G, a, a' \in A$.

Rmk $g0 = 0$, $g(-a) = -ga$

Ex Any abelian group A with the <u>trivial G-action</u>:
$ga = a$ $\forall g, a$.
(We equip $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$ with the trivial G-action unless otherwise stated.)

Ex $L|K$ fin. Gal. ext., $G = \mathrm{Gal}(L|K)$
$\leadsto$ G-modules L, $L^\times$, $\mu_n(L^\times) = \{x \in L^\times \mid x^n = 1\}$
$\mathcal{O}_L | \mathcal{O}_K$ corr. ext. of Ded. dom. $\leadsto$ $\mathcal{O}_L$, $\mathcal{O}_L^\times$
$L|K$ number fields $\leadsto$ $J(L) = \{\text{frac. id. of } L\}$, Cl_L
E elliptic curve over K $\leadsto$ $E(L)$
$\vdots$

Def A hom. of G-modules is a hom. $f: A \to B$ of groups s.t. $f(ga) = g f(a)$ $\forall g \in G, a \in A$.

Def Construct G-modules $A \times B$, A/B (for $B \subseteq A$ any sub-G-module), ... in the obvious way.

Rmk A (left) G-mod. A is the same as a left $\mathbb{Z}[G]$-module, where $\mathbb{Z}[G]$ is the group ring of G: The ring of formal sums $\sum_{g\in G} a_g \cdot g$ with $a_g \in \mathbb{Z}$ $\forall g \in G$

$a_g = 0$ for a.a $g \in G$.
(all but finitely many)

$$\sum_g a_g g + \sum_g b_g g = \sum_g (a_g + b_g) g$$

$$\left(\sum_g a_g g\right)\left(\sum_g b_g g\right) = \sum_{g,h} a_g b_h \, gh$$

$$= \sum_{i\in G} \underbrace{\left(\sum_{\substack{g,h\in G:\\ gh=i}} a_g b_h\right)}_{\in \mathbb{Z}} \cdot \underset{\in G}{i}$$

We'll often consider the "norm element"

$$N = N_G = \sum_{g\in G} g \in \mathbb{Z}[G].$$

Def The group of invariants is

$$A^G = \{a \in A \mid ga = a \ \forall g \in G\}$$ (= biggest subgroup of A with trivial G-action).

The group of co-invariants is

$$A_G = A / \langle ga - a \mid g \in G, a \in A \rangle$$ (= biggest quotient group of A with trivial G-action).

Ex $\mathbb{Z}^G = \mathbb{Z}$, $\mathbb{Z}_G = \mathbb{Z}$, $N_G \cdot x = \sum_{g \in G} g\cancel{x} = |G| \cdot x$

$(x \in \mathbb{Z})$

$L^G = K$, $L_G \cong K$, $N_G \cdot x = \sum_{g \in G} gx = \mathrm{Tr}_{L|K}(x)$
by the normal basis theorem

$(x \in L)$

$(L^\times)^G = K^\times$, $N_G \cdot x = \prod_{g \in G} gx = \mathrm{Nm}_{L|K}(x)$

$(x \in L^\times)$

$J(L)^G \supseteq J(K)$

"$\not=$" iff $L|K$ is ramified at a prime
$\mathfrak{p} = (\mathfrak{P}_1 \cdots \mathfrak{P}_r)^e$
$\Rightarrow \mathfrak{P}_1 \cdots \mathfrak{P}_r \in J(L)^G$
$\notin J(K)$

8.2. Motivation

Lemma If $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is an exc. seq. of G-mod., we get exc. seq.
$0 \to A^G \xrightarrow{i} B^G \xrightarrow{p} C^G$
and $A_G \xrightarrow{i} B_G \xrightarrow{p} C_G \to 0$

Pf straightforward. $\square$

Ex $L|K$ gal. ext. of local fields (nonarchimedean

$1 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{v_K} \frac{1}{e}\mathbb{Z} \to 0$

$1 \to \mathcal{O}_K^\times \to K^\times \xrightarrow{v_K} \frac{1}{e}\mathbb{Z}$

surjective if and only if $e=1$ ($L|K$ unramified)

Ex $G = \{e, \sigma\}$ cyclic group of order 2

$\tilde{\mathbb{Z}}$ = group $\mathbb{Z}$ with nontriv. G-action: $ex = x$, $\sigma x = -x$ $\forall x \in \mathbb{Z}$

triv. G-action because $1 = -1$ in $\mathbb{Z}/2\mathbb{Z}$

$$0 \to \tilde{\mathbb{Z}} \xrightarrow{\cdot 2} \tilde{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$$0 \to 0 \xrightarrow{\cdot 2} 0 \xrightarrow[\text{not surj.}]{} \mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow[\text{not inj.}]{\cdot 2} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Questions
- How nonsurjective is $B^G \to C^G$?
- How to tell if a given element of C^G lies in the image of B^G?

Def Let $C^1(G,A) = \{(a_g)_{g\in G} \mid a_g \in A \; \forall g \in G\}$ (group of 1-cochains)

$\cup\!\shortmid$

$\boxed{(*)}$ $Z^1(G,A) = \{(a_g)_{g\in G} \mid a_{gh} = a_g + g a_h\}$ (group of 1-cocycles)

$gha - a = ga - a + g(ha - a)$

$\cup\!\shortmid$

$B^1(G,A) = \{(ga - a)_{g \in G} \mid a \in A\}$ (group of 1-coboundaries)

$H^1(G,A) = Z^1(G,A)/B^1(G,A)$ (first cohomology group)

Rmk (Functoriality in A)

Any hom. $A \to B$ of G-modules induces a hom. of $H^1(G,A) \to H^1(G,B)$ of groups.

($H^1(G, \cdot)$ is a functor $\{G\text{-mod.}\} \to \{\text{ab. gr.}\}$.)

Ex If G acts trivially on A, then

$B^1(G,A)=0$

$\Rightarrow H^1(G,A)=Z^1(G,A)=\{(a_g)_{g\in G} \mid a_{gh}=a_g+\cancel{g}a_h \ \forall g,h\}$

$= \mathrm{Hom}_{\text{group}}(G,A)$

$(f_1+f_2)(g)=f_1(g)+f_2(g)$
for $f_1,f_2\in \mathrm{Hom}_{gr}(G,A)$

Thm If $0\to A\xrightarrow{i} B\xrightarrow{p} C\to 0$ an ex. seq. of G-mod., we get an ex. seq. of groups

$$0\to A^G\xrightarrow{i} B^G\xrightarrow{p} C^G \xrightarrow{\delta} H^1(G,A)\xrightarrow{i} H^1(G,B)\xrightarrow{p} H^1(G,C)$$

Pf w.l.o.g. $A\underset{i}{\subseteq} B$ sub-G-module, $C\underset{p}{=} B/A$.

Def of δ: For any $c\in C^G$, choose $b\in B$ s.t. $(b \bmod A)=c$.

$(gb-b \bmod A)=g(b \bmod A)-(b \bmod A)$

$= gc-c \underset{\substack{\uparrow \\ c\in C^G}}{=} 0 \qquad \forall g\in G.$

$\Rightarrow gb-b\in A \quad \forall g\in G$

$\Rightarrow (gb-b)_{g\in G}\in C^1(G,A)$

$\overset{(*)}{\Longrightarrow} (gb-b)_{g\in G}\in Z^1(G,A)$

b is unique mod A. $\Rightarrow (gb-b)_{g\in \cancel{A}}$ is unique mod $B^1(G,A)$.

$\rightsquigarrow \delta(c):=\left((gb-b)_{g\in G} \bmod B^1(G,A)\right)\in H^1(G,A)$

is a well-def. el. of $H^1(G,A)$ indep. of the choice of b.

δ hom.: clear

$b \in B^G \Rightarrow \delta(b \bmod A) = 0$: $\quad gb - b = 0 \;\; \forall g \in G$

$c \in C^G$, $\delta(c) = 0 \Rightarrow \exists\, b \in B^G : (b \bmod A) = c$:

$$\delta(c) = 0 \Rightarrow \exists\, b \in B : (b \bmod A) = c, \quad \underbrace{(gb-b)_{g\in G} = 0}_{\Downarrow \; b \in B^G}$$

Rest is similarly easy diagram chasing... $\square$

(This proof is the motivation for the def. of $H^1(G,A)$!)

Cor If $H^1(G,A) = 0$, then $B^G \to C^G$ is surjective.

$(0 \to A^G \to B^G \to C^G \to 0)$

depends only on A, not on B, C!

$c \in C^G \rightsquigarrow$ choose any b: $(b \bmod A) = c$

Q: $\exists\, a \in A$: $\underbrace{b + a \in B^G}$?

$\forall g \in G: (gb - b) + (ga - a) = 0$

$\Updownarrow$

$(gb-b)_{g\in G} + (ga-a)_{g\in G} = 0$

$\rightsquigarrow$ def. of 1-coboundaries