Def A hom. $f: F \to G$ of formal groups over R is a power series $f \in R[[X]]$ with $f(0) = 0$ and $f(F(X,Y)) = G(f(X), f(Y))$.

Def/Lemma $\mathrm{End}_R(F) = \{ f : F \to F \text{ hom.} \}$ is a ring with

addition $(f+g)(X) = F(f(X), g(X))$. $\triangledown!$

multiplication $(f \circ g)(X) = f(g(X))$.

7.2. Formal modules

Def A formal R-module F is a formal group F over R together with a ring hom.

$$R \to \mathrm{End}_R(F)$$
$$a \mapsto [a]_F(X)$$

satisfying

$$[a]_F(X) = aX + (\deg. \geq 2) \quad \forall a \in R$$

($\approx$ mult. by a close to 0).

Def A hom. $f: F \to G$ of formal R-modules is a hom. of formal groups s.t.

$$f([a]_F(X)) = [a]_G(f(X)) \quad \forall a \in R.$$

Ex $[a]_{G_a}(X) = aX$ (trivial additive R-module).

7.3. Lubin-Tate modules

Let K be a nonarch. local field with res. field $\mathbb{F}_q$.

Def A Lubin-Tate series for a uniformizer π is a power series $e \in \mathcal{O}_K[[X]]$ s.t.

i) $e(X) = \pi X + (\deg. \geq 2)$

ii) $e(X) \equiv X^q \mod \mathfrak{p}_K$.

Ex $e(X) = X^q + \pi X$.

Ex $f(X) = X^{q-1} + \dots + \pi$ monic Eisenstein pol. of degree $q-1$.

$\Rightarrow e(X) = X \cdot f(X)$ is a L-T series for π.

Ex $K = \mathbb{Q}_p$, $\pi = p$

$\rightsquigarrow e(X) = (X+1)^p - 1 = X^p + pX^{p-1} + \dots + pX$.

Thm A Let $e(X)$ be a L-T series for π. There is a unique formal $\mathcal{O}_K$-module F_e (the Lubin-Tate module for e) s.t. $[\pi]_{F_e}(X) = e(X)$.

Ex $K = \mathbb{Q}_p$, $\pi = p$, $e(X) = (X+1)^p - 1$.

$\rightsquigarrow F_e(X,Y) = \mathbb{G}_m(X,Y) = (X+1)(Y+1) - 1$

$[a]_F(X) = (X+1)^a - 1 = \sum_{i=1}^{\infty} \binom{a}{i} X^i$ for $a \in \mathbb{Z}_p$

$\left(\binom{a}{i} = \frac{a \cdots (a-i+1)}{i!} \right)$

Thm B If $e(X), \tilde{e}(X)$ are L-T series for the same π, then $F_e, F_{\tilde{e}}$ are isomorphic formal $\mathcal{O}_K$-modules.

$\rightsquigarrow F_\pi := F_e$.

The Thm follows from the following Lemma.

Lemma Let $e(X), \tilde{e}(X)$ be L-T series for π and let $a_1, \dots, a_r \in \mathcal{O}_K$. Then, there is exactly one power series $\phi \in \mathcal{O}_K[[X_1, \dots, X_r]]$ s.t.

- $\phi(X_1, \dots, X_r) = a_1 X_1 + \dots + a_r X_r + (\deg. \geq 2)$
- $e(\phi(X_1, \dots, X_r)) = \phi(\tilde{e}(X_1), \dots, \tilde{e}(X_r))$.

Pf of Thm A using the lemma

There is a unique $F_e(X,Y) = X + Y + (\deg. \geq 2)$ s.t.

$$e(F_e(X,Y)) = F_e(e(X), e(Y)).$$

There is a unique $[a]_{F_e}(X) = aX + (\deg. \geq 2)$ s.t.

$$e([a]_{F_e}(X)) = [a]_{F_e}(e(X)).$$

We need to show:

$$F_e(X,Y) = F_e(Y,X)$$
$$F_e(X, F_e(Y, Z)) = F_e(F_e(X,Y), Z)$$
$$[a]_{F_e}(F_e(X,Y)) = F_e([a]_{F_e}(X), [a]_{F_e}(Y))$$
$$[a+b]_{F_e}(X) = F_e([a]_{F_e}(X), [b]_{F_e}(X))$$
$$[ab]_{F_e}(X) = [a]_{F_e}([b]_{F_e}(X))$$
$$[1]_{F_e}(X) = X$$
$$[\pi]_{F_e}(X) = e(X).$$

The statements follow from the uniqueness claim in the lemma. For example:

- $F_e(X, F_e(Y,Z))$ and $F_e(F_e(X,Y), Z)$ are both the power series $\phi(X,Y,Z) = X+Y+Z+(\deg.\geq 2)$ such that $e(\phi(X,Y,Z)) = \phi(e(X),e(Y),e(Z))$.

- $[a]_{F_e}(F_e(X,Y))$ and $F_e([a]_{F_e}(X),[a]_{F_e}(Y))$ are both the power series $\Phi(X,Y) = aX+aY+(\deg.\geq 2)$ such that $e(\phi(X,Y)) = \phi(e(X),e(Y))$.

...

$\square$

Pf of Thm B

similar. $\square$

Pf of Lemma Write $X=(X_1,\dots,X_r)$, $e(X)=(e(X_1),\dots,e(X_r))$.

Write $\phi(X) = \phi_1(X)+\phi_2(X)+\dots$ with $\phi_n(X)\in \mathcal{O}_K[X]$ homogeneous of degree n. We inductively construct ϕ_n so that $e(\phi_1(X)+\dots+\phi_n(X)) = \phi_1(\tilde{e}(X))+\dots+\phi_n(\hat{e}(X))+(\deg.\geq n+1)$, starting with $\phi_1(X) = a_1X_1+\dots+a_rX_r$.

(Note that
$$\begin{aligned} e(\phi_1(X)) &\overset{i)}{=} \pi\,\phi_1(X)+(\deg.\geq 2 \text{ in } \phi_1(X)) \\ &= \pi(a_1X_1+\dots+a_rX_r)+(\deg.\geq 2) \\ &\overset{i)}{=} a_1\hat{e}(X_1)+\dots+a_r\hat{e}(X_r)+(\deg.\geq 2) \\ &= \phi_1(\tilde{e}(X))+(\deg.\geq 2).\,) \end{aligned}$$

Assume we have constructed $\phi_1,\dots,\phi_{n-1}$.

For any ϕ_n (hom. deg. n), we

$$e(\phi_1(X)+\dots+\phi_n(X)) \underset{i)}{=} e(\phi_1(X)+\dots+\phi_{n-1}(X)) + \pi\,\phi_n(X) + (\deg \geq n+1)$$

$$\phi_1(\tilde{e}(X)) + \dots + \phi_n(\underbrace{\tilde{e}(X)}_{\pi X + (\deg \geq 2)}) = \phi_1(\tilde{e}(X)) + \dots + \phi_{n-1}(\tilde{e}(X)) + \pi^n \phi_n(X) + (\deg \geq n+1)$$

This forces us to take $\phi_n :=$ hom. deg. n part of

$$\frac{e(\phi_1(X)+\dots+\phi_{n-1}(X)) - (\phi_1(\tilde{e}(X))+\dots+\phi_{n-1}(\tilde{e}(X)))}{\pi^n - \pi}.$$

It remains to show that the coefficients lie in $\mathcal{O}_K$, in other words that the numerator is divisible by π. (Because $\pi^n - \pi$ is divisible by $\mathfrak{p}$ exactly once.)

But
$$\begin{aligned} e(\phi_1(X)+\dots+\phi_{n-1}(X)) &\underset{ii)}{\equiv} (\phi_1(X)+\dots+\phi_{n-1}(X))^q \\ &\equiv \phi_1(X)^q + \dots + \phi_{n-1}(X)^q \\ &\equiv \phi_1(X^q) + \dots + \phi_{n-1}(X^q) \\ &\underset{ii)}{\equiv} \phi_1(\tilde{e}(X)) + \dots + \phi_{n-1}(\tilde{e}(X)) \mod \mathfrak{p} \end{aligned}$$

$(\cdot)^q$ is a hom. mod $\mathfrak{p}$

$t \equiv t^q \mod \mathfrak{p}$ $\forall t \in \mathcal{O}_K$

$\square$

7.4. Turning formal into ordinary groups/modules

Let K be a nonarch. local field.

If F is a formal group over $\mathcal{O}_K$, then for any $x, y \in \mathfrak{p}_K$,

$F(x,y) = x + y + (\deg. \geq 2)$ converges in $\mathfrak{p}_K$.

$\rightsquigarrow$ We obtain a group operation $\underset{F}{+}$ on $\mathfrak{p}_K$ with the identity $0 \in \mathfrak{p}_K$. $\quad (x \underset{F}{+} y = F(x,y))$

If F is a formal $\mathcal{O}_K$-module, then for any $a \in \mathcal{O}_K$, $x \in \mathfrak{p}_K$,

$[a]_F(x) = ax + (\deg. \geq 2)$ converges in $\mathfrak{p}_K$.

$\rightsquigarrow$ We obtain a scalar mult. operation $\underset{F}{\bullet}$ by el. of $\mathcal{O}_K$.

$$(a \underset{F}{\bullet} x = [a]_F(x)).$$

Similarly, formal hom. of formal groups/modules can be turned into actual (ordinary) hom. of ordinary groups/modules.

Ex $\quad x \underset{\mathbb{G}_a}{+} y = x + y \quad \rightsquigarrow$ group $(\mathfrak{p}_K, +)$

$\quad a \underset{\mathbb{G}_a}{\bullet} x = ax$

Ex $\quad x \underset{\mathbb{G}_m}{+} y = (x+1)(y+1) - 1 \rightsquigarrow$ group $\cong (\underset{1+\mathfrak{p}_K}{\underset{\|}{U_K^{(1)}}}, \cdot)$

In fact, the power series converge for any elements of $\mathfrak{p}_{\bar{K}}$.

(Reduce to finite extensions of K, which are all complete.)