6.3 Examples

Some totally ramified extensions $L \mid K$:

<u>Ex</u> $\mathbb{Q}_p(\sqrt{p}) \mid \mathbb{Q}_p \qquad (p \neq 2)$

$\mathbb{Z}_p[\sqrt{p}] \mid \mathbb{Z}_p$

$\mathrm{Gal} = \{id, \sigma\} = \mathbb{Z}/2$

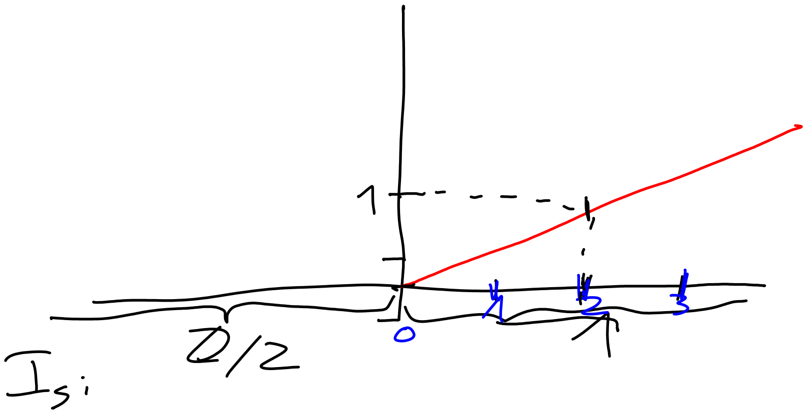
$$i(\sigma) = v_L(\underbrace{\sigma(\sqrt{p})}_{-\sqrt{p}} - \sqrt{p}) = v_L(-2\sqrt{p}) \overset{\text{tot. ram.}}{=} v_K(N_{L|K}(-2\sqrt{p})) = v_K(4p) = 1.$$

$I_0 = \mathbb{Z}/2 = I^0$

$I_1 = 1$

$I_2 = 1 = I^1$

$\vdots$



<u>Ex</u> $\mathbb{Q}_2(\sqrt{p}) \mid \mathbb{Q}_2 \qquad (p \equiv 3 \mod 4)$

$\mathbb{Z}_2[\sqrt{p}] \mid \mathbb{Z}_2$

$$i(\sigma) = v_L(-2\sqrt{p}) = v_K(4p) = 2$$

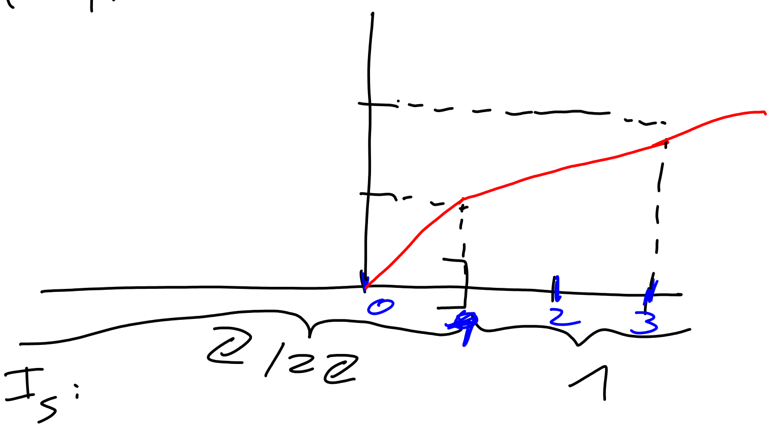
$I_0 = \mathbb{Z}/2 = I^0$

$I_1 = \mathbb{Z}/2 = I^1$

$I_2 = 1$

$I_3 = 1 = I^2$

$\vdots$



<u>Ex</u> $\mathbb{Q}_2(\sqrt{2}) \mid \mathbb{Q}_2$

$\mathbb{Z}_2[\sqrt{2}] \mid \mathbb{Z}_2$

$$i(\sigma) = v_K(8) = 3$$

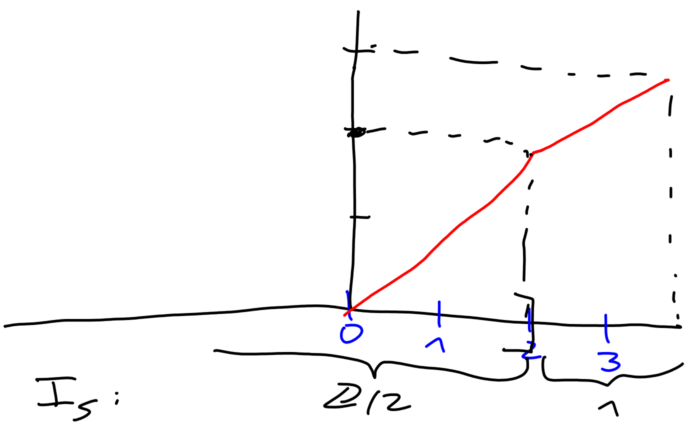
$I_0 = \mathbb{Z}/2 = I^0$

$I_1 = \mathbb{Z}/2 = I^1$

$I_2 = \mathbb{Z}/2 = I^2$

$I_3 = 1$

$\vdots$



Ex $K_n := \mathbb{Q}_p(\zeta_{p^n}) | \mathbb{Q}_p$ tot. ram. of degree $p^{n-1}(p-1) = \varphi(p^n)$.

$\mathbb{Z}_p[\zeta_{p^n}] | \mathbb{Z}_p$

$\phi_r \longleftrightarrow r \bmod p^n$

$Gal(K_n | \mathbb{Q}_p) = (\mathbb{Z}/p^n\mathbb{Z})^\times$

$\cup$ $\cup$

$Gal(K_n | K_m) = \{ r \in (\mathbb{Z}/p^n\mathbb{Z})^\times | r \equiv 1 \bmod p^m \}$ $(m \leq n)$

$\zeta_{p^n} - 1$ is a uniformizer

Let $1 \neq r \in (\mathbb{Z}/p^n\mathbb{Z})^\times$.

$$i_{K_n|\mathbb{Q}_p}(\phi_r) = v_{K_n}(\phi_r(\zeta_{p^n} - 1) - (\zeta_{p^n} - 1))$$

$$= v_{K_n}(\zeta_{p^n}^r - \zeta_{p^n})$$

$$= v_{K_n}(\zeta_{p^n}^{r-1} - 1)$$

($\zeta_{p^n} \in \mathcal{O}_{K_n}^\times$)

$$= v_{K_n}(\zeta_{p^n}^{p^t} - 1) \quad \text{if } t = v_p(r-1)$$

($p^t = u(r-1)$, $u \in (\mathbb{Z}/p^n\mathbb{Z})^\times$)

$= \text{largest } t \leq n$ s.t. $\phi_r \in Gal(K_n|K_t)$

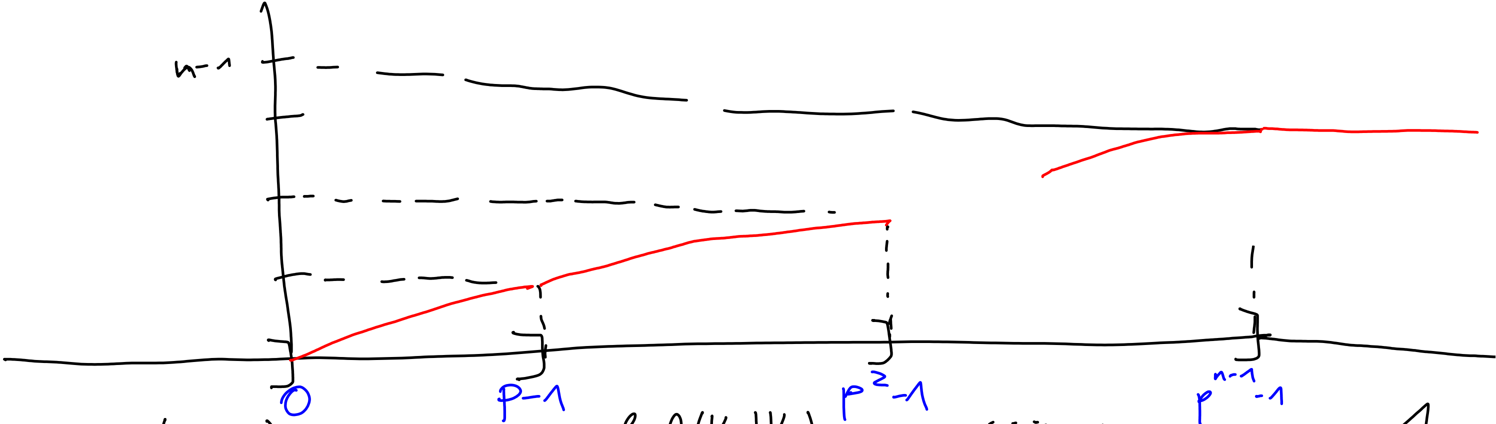
$$= v_{K_n}(\zeta_{p^{n-t}} - 1)$$

$$= \underbrace{[K_n : K_{n-t}]}_{p^t} \cdot \underbrace{v_{K_{n-t}}(\overbrace{\zeta_{p^{n-t}} - 1}^{\text{uniformizer in } K_{n-t}})}_{1}$$

(tot. ram.)

$$= p^t$$

$\Rightarrow I_s(K_n | \mathbb{Q}_p) = Gal(K_n | K_t)$, where t is the smallest $t \geq 0$ s.t. $s \leq p^t - 1$ or $t = n$.



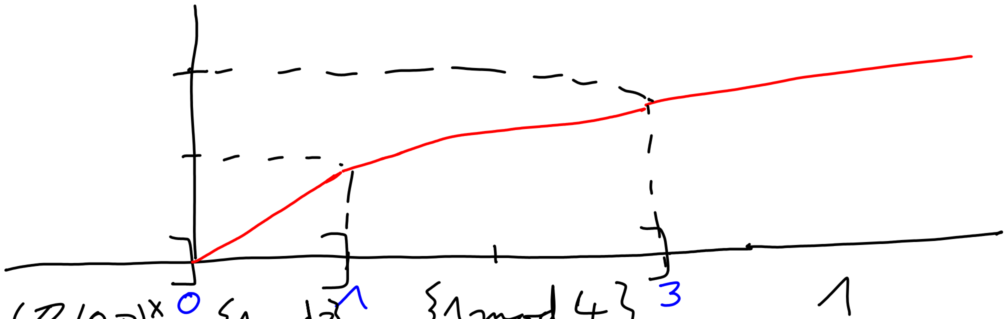
$Gal(K_n|\mathbb{Q}_p)$ $\quad Gal(K_n|K_1)$ $\quad Gal(K_n|K_2)$ $\quad \cdots Gal(K_n|K_{n-1})$ $\quad 1$

$$I^t(K_n|\mathbb{Q}_p) = Gal(K_n|K_t)$$

Ex of Ex $(p^n = 8)$

$\mathbb{Q}_2(\sqrt{2}) = \mathbb{Q}_2(\zeta_8)^{\{\pm 1\}}$

$\sqrt{2} = \zeta_8 + \zeta_8^{-1}$



I_s: $(\mathbb{Z}/8\mathbb{Z})^\times = \{1 \bmod 2\}$ $\quad \{1 \bmod 4\}$ $\quad 1$

$I_s \bmod \{\pm 1\}$: $\underbrace{\qquad\qquad}_{\mathbb{Z}/2\mathbb{Z}}$ $\quad 1$

$\mathbb{Q}_2(\zeta_8)$
|
$\mathbb{Q}_2(\sqrt{2})$
|
$\mathbb{Q}_2$

$I^0 = (\mathbb{Z}/8\mathbb{Z})^\times \qquad I^3 = 1$

$I^1 = (\mathbb{Z}/8\mathbb{Z})^\times$

$I^2 = \{1 \bmod 4\}$

6.4. Upper numbering

Def $\eta_{L|K}(s) := \int_0^s \frac{dx}{[I_0 : I_x]} = \frac{1}{|I_0|} \cdot \sum_{\sigma \in Gal(L|K)} \overbrace{\min(i_{L|K}(\sigma), s+1)}^{\mathbb{Z}} - 1$

For $s = 0$: $i_{L|K}(\sigma) \geq 1 \Leftrightarrow \sigma \in I_0$

$\frac{d}{ds}$ agrees: $i_{L|K}(\sigma) \geq s+1 \Leftrightarrow \sigma \in I_s$

The t-th ramification group (in upper numbering) is $I^t(L|K) = I_{\eta^{-1}_{L|K}(t)}(L|K)$.

Thm (Herbrand)

If $F|K$ is a Galois subext., then $I^t(F|K)$ is the image of $I^t(L|K)$ under the restriction map $\mathrm{Gal}(L|K) \twoheadrightarrow \mathrm{Gal}(F|K)$.

$$\begin{array}{c} L \\ \text{Gal} \mid \quad \\ F \\ \text{Gal} \mid \quad \\ K \end{array} \quad \text{(Gal.)}$$

Pf Neukirch, chapter II.10. □

Cor The following def. is consistent:

Def For any Gal. ext. $L|K$, let

$$I^t(L|K) = \{\sigma \in \mathrm{Gal}(L|K) \mid \forall F|K \text{ finite gal. ext.}\ (F \subseteq L) : \sigma|_F \in I^t(L|K)\}.$$

Ex

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p) \cong \widehat{\mathbb{Q}_p^\times} \cong \mathbb{Z}_p^\times \times \hat{\mathbb{Z}}$$
(← profinite completion)

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p^{unram}(\zeta_{p^t})) \cong U^{(t)}_{\mathbb{Q}_p}$$
$$= I^t(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p)$$

$$I^0(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p) \cong U^{(0)} = \mathbb{Z}_p^\times$$
$$\cup \qquad\qquad \cup$$
$$I^1(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p) \cong U^{(1)} = 1 + p\mathbb{Z}_p$$
$$\cup \qquad\qquad \cup$$
$$I^2(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p) \cong U^{(2)} = 1 + p^2\mathbb{Z}_p$$
$$\cup \qquad\qquad \cup$$
$$\dots \qquad\qquad \dots$$

6.5. Abelian extensions

Thm (Hasse–Arf) If $L|K$ is abelian, then the "corners" of $\eta_{L|K}$ have integer coordinates.

In other words, $\forall t \in \mathbb{R}^{>0} \setminus \mathbb{Z} \ \exists \varepsilon > 0 : I^t(L|K) = I^{t+\varepsilon}(L|K)$

"I^t only changes at integers t."

Pf Serre, Local field, chapter V. $\square$

Connection with CFT:

Property 6 of Artin reciprocity Let K be a local field.

Then, $U_K^{(t)} = \theta_K^{-1}(I^t(K^{ab}|K))$ for any $t \in \mathbb{Z}^{\geq 0}$.

$$
\begin{array}{ccc}
K^\times & \xrightarrow{\theta_K} & \mathrm{Gal}(K^{ab}|K) \\
\cup & & \cup \\
\mathcal{O}_K^\times = U_K^{(0)} & \longrightarrow & I^0 \\
\cup & & \cup \\
U_K^{(1)} & \longrightarrow & I^1 \\
\cup & & \cup \\
I_K^{(2)} & \longrightarrow & I^2 \\
\cup & & \cup \\
\vdots & & \vdots
\end{array}
$$

Cor $I^t(K^{ab}|K) / I^{t+1}(K^{ab}|K) \cong U_K^{(t)} / U_K^{(t+1)}$

$$\cong \begin{cases} k_K^\times, & t = 0 \\ k_K, & t \geq 1. \end{cases}$$

for any $t \in \mathbb{Z}^{\geq 0}$.

Rmk Klasse-Krf $\Longrightarrow$ Local Kronecker-Weber

Pf Let $K|\mathbb{Q}_p$ be a finite abelian ext.

Let $I^t(K|\mathbb{Q}_p)=1$.

Let $K' = K \cap \mathbb{Q}_p^{unram}$ $\left(\subseteq \mathbb{Q}_p(\zeta_\infty)\right)$.

K
| tot. ram.
K'
| unram
$\mathbb{Q}_p$

Goal: $K \subseteq K'(\zeta_{p^t})$.

Recall that $I^t(\mathbb{Q}_p(\zeta_{p^t})|\mathbb{Q}_p)=1$.

$\rightsquigarrow$ W.l.o.g. $K \supseteq K'(\zeta_{p^t})$.

(replace K by $K(\zeta_{p^t}) = K\cdot K'(\zeta_{p^t})$)

$$[K:K'] = |I(K|\mathbb{Q}_p)| = |I^0/I^1(K|\mathbb{Q}_p)|\cdot|I^1/I^2|\cdots|I^{t-1}/I^t|$$

(Lemma in 6.1)

$$\leq |\mathbb{F}_p^\times|\cdot|\mathbb{F}_p|\cdots|\mathbb{F}_p|$$

$$= |I^0/I^1(\mathbb{Q}_p(\zeta_{p^t})|\mathbb{Q}_p)|\cdot|I^1/I^2|\cdots|I^{t-1}/I^t|$$

$$= |I(\mathbb{Q}_p(\zeta_{p^t})|\mathbb{Q}_p)|$$

$$\overset{K'|\mathbb{Q}_p \text{ unram.}}{=} |I(K'(\zeta_{p^t})|K')|$$

$$= [K'(\zeta_{p^t}):K']$$

$$\Longrightarrow K = K'(\zeta_{p^t}) \subseteq \mathbb{Q}_p(\zeta_\infty).$$ $\square$

More generally:

<u>Thm</u> Let $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$ be abelian ext. of a local field K with residue field $\mathbb{F}_q$ such that

$K_0 = K^{unram}$, $I^u(K_n|K) = 1$,

$$[K_{n+1} : K_n] = \begin{cases} q-1, & n=0 \\ q, & n \geq 1 \end{cases}.$$

Then, $K^{ab} = \bigcup_{n \geq 0} K_n$.

<u>Construction</u>

The following construction turns out to work:

Let $f(X) \in \mathcal{O}_K[X]$ be an Eisenstein polynomial of degree $q-1$ and let $e(X) = X \cdot f(X)$. Let

α_1 be a root of $f(X)$,

α_2 — " — $f(e(X))$,

α_3 — " — $f(e(e(X)))$,

⋮

Let $K_{\pi,n} = K(\alpha_n)$ depends only on the uniformiser $\pi = f(0)$ and n, and we can take $K_n = K^{unram} \cdot K_{\pi,n} = K^{unram}(\alpha_n)$.

$$\Rightarrow K^{ab} = \bigcup_{n \geq 0} K_n.$$

<u>Ex</u> If $K = \mathbb{Q}_p$ with $e(X) = (X+1)^p - 1 = X^p + pX^{p-1} + \dots + pX$, we get $\alpha_n = \zeta_{p^n} - 1$,

$K_{p,n} = \mathbb{Q}_p(\zeta_{p^n})$, $K_n = \mathbb{Q}_p^{unram}(\zeta_{p^n})$.