Def $\mathfrak{P}|\mathfrak{p}$ is <u>unramified</u> if $I_0(\mathfrak{P}|\mathfrak{p}) = 1$.

$\mathfrak{P}|\mathfrak{p}$ is <u>tamely ramified</u> if $I_1(\mathfrak{P}|\mathfrak{p}) = 1$.

Lemma $I_s(\mathfrak{P}|\mathfrak{p})$ is a normal subgroup of $D(\mathfrak{P}|\mathfrak{p})$.

Lemma If $F|K$ is a subext. of $L|K$, then

$$\begin{tikzcd} L \arrow[d, dash] & \mathfrak{P} \arrow[d, dash] \\ F \arrow[d, dash] & P \arrow[d, dash] \\ K & \mathfrak{p} \end{tikzcd} \qquad I_s(\mathfrak{P}|P) = I_s(\mathfrak{P}|\mathfrak{p}) \cap Gal(L|F).$$

Rmk If K is a global field, then

$$Gal(L_{\mathfrak{P}}|K_{\mathfrak{p}}) = D(\mathfrak{P}|\mathfrak{p})$$
$$I_s(L_{\mathfrak{P}}|K_{\mathfrak{p}}) = I_s(\mathfrak{P}|\mathfrak{p}) \qquad \forall s \geq 0.$$

$\Rightarrow$ "Often, we can reduce to ext. of local fields."

Rmk If $\mathcal{O}_L = \mathcal{O}_K[a_1, \dots, a_t]$, it suffices to consider only $a = a_1, \dots, a_t$ in the def. of I_s and $i_{L|K}$.

Lemma $L|K$ fin. gal. ext. of local fields

$$\begin{aligned} I_s(L|K) &= \{\sigma \in \underline{\underline{I}}(L|K) \mid \sigma(\pi_L) \equiv \pi_L \bmod \mathfrak{P}_L^{s+1}\} \\ &= \{\sigma \in I(L|K) \mid \tfrac{\sigma(\pi_L)}{\pi_L} \equiv 1 \bmod \mathfrak{P}_L^s\} \\ &= \{\sigma \in I(L|K) \mid \tfrac{\sigma(\pi_L)}{\pi_L} \in U_L^{(s)}\} \end{aligned}$$

and $i_{L|K}(\sigma) = v_L(\sigma(\pi_L) - \pi_L)$ if $\sigma \in \underline{\underline{I}}(L|K)$.

Pf Let $F = L^{I(L|K)} = L \cap K^{unram}$ be the max. unram. subext.

$$\Rightarrow I_s(L|K) = I_s(L|F) = \{ \sigma \in \underbrace{Gal(L|F)}_{I(L|K)} \mid \sigma(\pi_L) \equiv \pi_L \bmod \mathfrak{p}_L^{s+1} \}$$

$O_L = O_F[\pi_L]$ according to a Thm. in section 1.6.

$$\begin{array}{l} L \\ \mid \text{tot. ram.} \\ F \\ \mid \text{unram.} \\ K \end{array}$$

$\square$

Cor We obtain injective group hom.

$$I_0/I_1 \hookrightarrow O_L^\times / U_L^{(1)} \cong k_L^\times$$

$$[x] \mapsto x \bmod \mathfrak{p}_L$$

$$I_s/I_{s+1} \hookrightarrow U_L^{(s)}/U_L^{(s+1)} \cong k_L \quad \text{for } s \geq 1.$$

$$[\sigma] \mapsto \left[\frac{\sigma(\pi_L)}{\pi_L}\right] \qquad [1+\pi_L^s y] \mapsto y \bmod \mathfrak{p}_L$$

depends on choice of π_L

indep. of choice of π_L.

Pf Well-def.: $\frac{\sigma(\pi_L)}{\pi_L} \in O_L^\times$. If $\sigma \in I_s$, then $\frac{\sigma(\pi_L)}{\pi_L} \in U_L^{(s)}$.

Indep. of π_L: Let $\sigma \in I_s$, $\alpha \in O_L^\times$, then $v_L(\sigma(\alpha) - \alpha) \geq s+1$, so $v_L\left(\frac{\sigma(\alpha)}{\alpha} - 1\right) \geq s+1$, so $\frac{\sigma(\alpha)}{\alpha} \in U_L^{(s+1)}$.

Hence, $\frac{\sigma(\pi_L)}{\pi_L} U_L^{(s+1)} = \frac{\sigma(\alpha\pi_L)}{\alpha\pi_L} U_L^{(s+1)}$.

Group hom. If $\sigma, \tau \in I_s$, then

$$\frac{\sigma\tau(\pi_L)}{\pi_L} \cdot U_L^{(s+1)} = \frac{\tau(\pi_L)}{\pi_L} \cdot \frac{\sigma(\tau(\pi_L))}{\tau(\pi_L)} \cdot U_L^{(s+1)}$$

$$= \frac{\tau(\pi_L)}{\pi_L} \cdot \frac{\sigma(\pi_L)}{\pi_L} \cdot U_L^{(s+1)}$$

($\tau(\pi_L)$ is also a uniformizer)

Injective: $\sigma \in I_{s+1} \iff \frac{\sigma(\pi_L)}{\pi_L} \in U_L^{(s+1)}$. $\square$

Summary Let $k_L = \mathbb{F}_{q^f}$, $k_K = \mathbb{F}_q$, $q = p^r$.

$$Gal(L|K) = D \supseteq I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$$

quot. $\mathbb{Z}/f\mathbb{Z}$

quot. $\subseteq k_L^\times = \mathbb{Z}/(q^f-1)\mathbb{Z}$ (size coprime to p)

quot. $\subseteq k_L = (\mathbb{Z}/p\mathbb{Z})^{rf}$ (size power of p)

Cor $Gal(L|K)$ is solvable

Cor $I_1(L|K)$ is the unique p-Sylow subgroup of $I(L|K)$.

all p-Sylow subgroups are conjugate, but $I_1(L|K)$ is normal

Cor $L|K$ is tamely ramified if and only if $p \nmid |I(L|K)|$.

Lemma If $L|K$ is abelian, we even get injective group hom.

$$I_0/I_1 \hookrightarrow k_K^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$$

$$I_s/I_{s+1} \hookrightarrow k_K \cong (\mathbb{Z}/p\mathbb{Z})^r \qquad \text{for } s \geq 1.$$

$(k_K = \mathbb{F}_q, \quad q = p^r)$.

Pf Let $\sigma \in I_s$, $x = \dfrac{\sigma(\pi_L)}{\pi_L} \in U_L^{(s)} / U_L^{(s+1)}$.

Let $\widetilde{\varphi_q} \in Gal(L|K)$ be a lift of the Frob. aut. φ_q.

$$\widetilde{\varphi_q}(x) = \frac{\widetilde{\varphi_q}\,\sigma(\pi_L)}{\widetilde{\varphi_q}(\pi_L)} \underset{\substack{Gal(L|K)\\ \text{abelian}}}{=} \frac{\sigma\,\widetilde{\varphi_q}(\pi_L)}{\widetilde{\varphi_q}(\pi_L)} \underset{\substack{\widetilde{\varphi_q}(\pi_L)\\ \text{is also a}\\ \text{uniformizer}}}{=} \frac{\sigma(\pi_L)}{\pi_L} = x \mod U_L^{(s+1)}$$

Case $s=0$: $\varphi_q(x \bmod \mathfrak{p}_L) = (\widetilde{\varphi_q}(x) \bmod \mathfrak{p}_L)$

$\Rightarrow (x \bmod \mathfrak{p}_L) \in \mathbb{F}_q^\times = k_K^\times$. ✓

Case $s \geq 1$: Write $x = 1 + \pi_L^s y$.

$\Rightarrow \widetilde{\varphi_q}(1 + \pi_L^s y) = 1 + \pi_L^s y \mod U_L^{(s+1)}$.

$\Rightarrow \widetilde{\varphi_q}(\pi_L^s y) \equiv \pi_L^s y \mod \mathfrak{p}_L^{s+1}$.

$\Rightarrow \dfrac{\widetilde{\varphi_q}(\pi_L^s)}{\pi_L^s} \cdot \underset{\substack{\rotatebox{90}{$\equiv$}\\ \varphi_q(y) = y^q}}{\widetilde{\varphi_q}(y)} \equiv y \mod \mathfrak{p}_L$.

This congruence has at most q sol. $y \in k_L$.

$\Rightarrow |\underbrace{\text{im of } I_s/I_{s+1} \text{ in } k_L}| \leq q = p^r$.

$\leq k_L = \mathbb{F}_{q^f} = (\mathbb{Z}/p\mathbb{Z})^{fr}$

$\Rightarrow \text{im} \leq (\mathbb{Z}/p\mathbb{Z})^r \cong \mathbb{F}_q$. $\square$

6.2. Discriminant formula

Thm $L|K$ fin. Gal. ext. of local fields

$$\Rightarrow v_K(\mathrm{disc}(L|K)) = f(L|K)\cdot \sum_{\mathrm{id}\neq\sigma\in \mathrm{Gal}(L|K)} i_{L|K}(\sigma)$$

$$= f(L|K)\cdot \sum_{s=0}^{\infty} \left(|I_s(L|K)|-1\right).$$

Lemma $L|K$ fin. ext. of local fields.

$\Rightarrow \mathcal{O}_L = \mathcal{O}_K[\alpha]$ for some $\alpha\in\mathcal{O}_L$.

Pf Let $\mathbb{F}_{q^f}|\mathbb{F}_q$ be the res. field ext.

$\Rightarrow \mathbb{F}_{q^f} = \mathbb{F}_q(\zeta_{q^f-1})$.

Hensel $\Rightarrow \zeta_{q^f-1}\in\mathcal{O}_L$

Let $\alpha = \zeta_{q^f-1} + \pi_L$.

$$\Rightarrow \beta := \alpha^{q^f} - \alpha = \cancel{\zeta_{q^f-1}^{q^f}} + \pi_L^{q^f} - \cancel{\zeta_{q^f-1}} - \pi_L$$

$$\equiv -\pi_L \mod \mathfrak{p}_L^2.$$

$\Rightarrow v_L(\beta) = 1.$ $\Rightarrow \beta$ is a uniformizer in L.

$\Rightarrow \mathcal{O}_K[\alpha]$ contains a uniformizer and a generator $(\alpha \bmod \mathfrak{p}_L) = \zeta_{q^f-1}$ of $\kappa_L|\kappa_K$.

$\Rightarrow \mathcal{O}_L = \mathcal{O}_K[\alpha]$. $\square$

Thm in section 1.6

Pf of Thm Let $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

$$\Rightarrow \operatorname{disc}(L|K) = \pm \prod_{\substack{\sigma,\tau \in \operatorname{Gal}(L|K) \\ \sigma \neq \tau}} (\sigma(\alpha) - \tau(\alpha))$$

$$= \pm \prod_{\sigma} \sigma\Big(\prod_{\tau \neq id} (\alpha - \tau(\alpha)) \Big).$$

$$\Rightarrow v_K(\operatorname{disc}(L|K)) = \frac{1}{e(L|K)} v_L(\operatorname{disc}(L|K))$$

$$= \frac{[L:K]}{e(L|K)} \sum_{\tau \neq id} v_L(\alpha - \tau(\alpha))$$

$$= f(L|K) \cdot \sum_{\tau \neq id} i_{L|K}(\tau) \qquad \square$$

Ex If $L|K$ is tamely ramified $(I_1 = 1)$, then

$$v_K(\operatorname{disc}(L|K)) = f(L|K) \cdot (e(L|K) - 1) = [L:K] - f(L|K).$$