Lemma 5.1 Let G be a commutative compact topological group such $\bigcap_{\substack{U \leq G \\ \text{open} \\ (\text{fin. index})}} U = 0$. Then $G = \hat{G}$.

$\uparrow$ profinite completion

Ex Let K be a number field.

$$\Rightarrow \hat{C}_K = C_K \Big/ \prod_{v \text{ real}} \mathbb{R}^{>0} \times \prod_{v \text{ complex}} \mathbb{C}^\times = C_K / (\text{conn. cp. of } C_K \text{ containing})$$

$$= \Big(\prod_{v \text{ nonarch}}' K_v^\times \times \prod_{v \text{ real}} \underbrace{\mathbb{R}^\times / \mathbb{R}^{>0}}_{\{\pm 1\}} \Big) \Big/ K^\times$$

Pf Consider the map $f : C_K \longrightarrow \hat{C}_K = \varprojlim_{\substack{U \leq C_K \\ \text{open,} \\ \text{finite index}}} C_K / U$

Recall the continuous inclusion $i_v : K_v^\times \hookrightarrow C_K$.

For any U and any v, the set $i_v^{-1}(U) = "U \cap K_v^\times" \subset K_v^\times$ is an open subgroup of $K_v^\times$.

$\Rightarrow$ For v real, $\mathbb{R}^{>0} \subseteq i_v^{-1}(U)$.

For v complex, $\mathbb{C}^\times = i_v^{-1}(U)$.

$\Rightarrow \prod_{v \text{ real}} \mathbb{R}^{>0} \times \prod_{v \text{ complex}} \mathbb{C}^\times \subseteq U \quad \forall U$

$\Rightarrow$ —— " —— $\subseteq \ker(f)$

In fact, —— " —— $= \bigcap_U U = \ker(f)$.

We also have a continuous surjective map

$$\underbrace{J_K^1}_{\{(x_v)_v \in \mathbb{A}_K^\times \mid \prod_v |x_v|_v = 1\}} / K^\times \twoheadrightarrow \mathbb{A}_K^\times / K^\times \cdot \Big(\prod_{v \text{ real}} \mathbb{R}^{>0} \times \prod_{v \text{ complex}} \mathbb{C}^\times\Big).$$

LHS is compact (Thm. in section 4.6)

$\Rightarrow$ RHS is compact.

By Lemma 5.1, f is surjective. $\square$

<u>Ese</u> Let K be a (global) function field.

$\Rightarrow$ $C_K \longrightarrow \widehat{C}_K$ is injective, but not surjective.

$$\underset{\text{not compact}}{\underset{\uparrow}{C_K}} \longrightarrow \underset{\text{compact}}{\underset{\uparrow}{\widehat{C}_K}}$$

$$\boxed{\begin{array}{ccc} C_K & \longrightarrow & \mathbb{R}^{>0} \\ \shortparallel & & \\ \mathbb{A}_K^\times / K^\times & & \\ (x_v)_v & \longmapsto & \prod_v |x_v|_v \end{array}}$$

Ex $K = \mathbb{Q}$

Kronecker-Weber: $\mathbb{Q}^{ab} = \mathbb{Q}(\zeta_\infty) = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_n)$

$$\mathrm{Gal}(\mathbb{Q}(\zeta_\infty)|\mathbb{Q}) = \hat{\mathbb{Z}}^\times = \prod \mathbb{Z}_p^\times = \left(\prod_p{}' \mathbb{Q}_p^\times \times (\mathbb{R}^\times/\mathbb{R}^{>0})\right) \Big/ \mathbb{Q}^\times$$

$$(\dots, p^{-1}, 1, p^{-1}, \dots) \quad p$$

$$(0,1) \mapsto (\dots, p^{-1}, 1, p^{-1}, \dots)$$

$$D(p) = \mathrm{Gal}(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p) = \mathbb{Z}_p^\times \times \hat{\mathbb{Z}} = \widehat{\mathbb{Q}_p^\times}$$

$$I(p) = \mathbb{Z}_p^\times$$

5.2. Hilbert class field

<u>Def</u> Let $U := \prod_{v \text{ nonarch}} \mathcal{O}_v^\times \times \prod_{v \text{ arch.}} K_v^\times \subseteq \mathbb{A}_K^\times$

The corr. field $K' := (K^{ab})^{\Theta_K(U)}$ is called the <u>Hilbert class field</u> of K.

<u>Ex</u> If $K = \mathbb{Q}$, then $K' = \mathbb{Q}$ because

$$\prod \mathbb{Z}_p^\times \times \mathbb{R}^\times \longrightarrow \prod{}' \mathbb{Q}_p^\times \times \mathbb{R}^\times / \mathbb{Q}^\times \text{ is surjective.}$$

<u>Thm</u> K' is the maximal abelian unram. ext. of K in which every arch (real) place splits completely (into real places).

<u>Pf</u> The field corr. to $U' \subseteq C_K$ is

- unramified at v if and only if $I(v) = \mathcal{O}_v^\times \subseteq U'$
- completely split at v if and only if $D(v) = K_v^\times \subseteq U'$.

$\square$

<u>Rmk</u> Some people (e.g. Milne) call $\mathbb{C}|\mathbb{R}$ ramified so they can say "K' is the max. unram. ext of K". But others (e.g. Neukirch) call $\mathbb{C}|\mathbb{R}$ unramified!

<u>Rmk</u> $\mathbb{Q}$ has no unramified field extensions (not even nonabelian ones).

Pf $K|\mathbb{Q}$ unramified $\iff D := \mathrm{disc}(K) = \pm 1$

Assume $n := [K:\mathbb{Q}] \geq 2$.

Minkowski's theorem implies that there exists some $0 \neq a \in \mathcal{O}_K$ such that

$$|N_{K|\mathbb{Q}}(a)| \leq \frac{n!}{n^n} \cdot \left(\frac{4}{\pi}\right)^{n/2} \cdot \sqrt{|D|}$$

$$= \frac{n!}{n^n} \cdot \left(\frac{4}{\pi}\right)^{n/2} < 1. \quad ↯ \qquad \square$$

Thm If K is a number field, then $\mathrm{Gal}(K'|K) \cong Cl_K$.

Pf $\mathrm{Gal}(K'|K) \cong (\mathbb{A}_K^\times / K^\times)/U = \mathbb{A}_K^\times / K^\times \cdot \left(\prod_{v \text{ nonarch}} \mathcal{O}_v^\times \times \prod_{v \text{ arch}} K_v^\times\right)$

$\cong Cl_K$. $\qquad \square$

Thm in section 4.6

Exe $K = \mathbb{Q}(\sqrt{-15}) \rightsquigarrow K' = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$

$$Cl_K = \left\{ \langle 1 \rangle, \langle (2, \tfrac{1+\sqrt{-15}}{2}) \rangle \right\} \cong \mathbb{Z}/2\mathbb{Z}$$
$$= \mathrm{Gal}(K'|K).$$

Rmk

$$\begin{array}{rll} & \vdots & \\ \text{unram.} & \Big| \; Cl_{K''} & \\ & K'' & \leftarrow \text{Hilbert class field of } K' \\ \text{unram.} & \Big| \; Cl_{K'} & \\ & K' & \\ \text{unram.} & \Big| \; Cl_{K} & \\ & K & \end{array}$$

Theorem (Golod-Shafarevich)

Sometimes, this tower is infinite ($Cl_{K''\cdots'} \neq 1$ after every step).

Ex: K imaginary quadratic extension of $\mathbb{Q}$ with $\mathrm{disc}(K)$ divisible by ≥ 6 different prime numbers.

Cor: Sometimes, K has an infinite (nonabelian) unramified extension.

[Reference: Cassels-Fröhlich.]

Thm (Principal ideal theorem)

Let K be a number field. Then, every ideal of K becomes principal in K'.

In other words, $Cl_K \longrightarrow Cl_{K'}$ is trivial.

__Ex__ $K = \mathbb{Q}(\sqrt{-15})$

$$\left(2, \frac{1+\sqrt{-15}}{2}\right) = \left(\frac{1+\sqrt{5}}{2}\right).$$

The thm follows from:

__Prop 5__ (Cofunctoriality)

For any fin. separable ext. $L|K$ of $\left\{\begin{matrix}\text{fin. local}\\ \text{global}\end{matrix}\right\}$ fields, we get a comm. diagram

$$\begin{tikzcd}
C_K \arrow[r, "\theta_K"] \arrow[d] & Gal(K^{ab}|K) = G^{ab} \arrow[d, "V"] \\
C_L \arrow[r, "\theta_L"'] & Gal(L^{ab}|L) = H^{ab}
\end{tikzcd}$$

where $V: G^{ab} \to H^{ab}$ is the __transfer__ (Verlagerung) map defined as follows. $(G = Gal(K^{sep}|K),\ H = Gal(K^{sep}|L))$

Def Let G be a compact top. group and let $H \leq G$ be an open (index n) subgroup. Let $g_1, \dots, g_n \in G$ be representatives of the cosets in $H \backslash G$. Then, define $V: G^{ab} \longrightarrow H^{ab}$:
For any $t \in G$, let $V(t) = \prod_{i=1}^{n} [h_i] \in H^{ab}$,
where we write $g_i t = h_i g_{\pi(i)}$
with $h_i \in H$, $\pi \in S_n$ some permutation.

Rmk V is a continuous hom. and does not depend on the choice of $g_1, \dots, g_n$.

Pf of the principal ideal theorem

$$\begin{tikzcd} K'' \arrow[d, dash, "Cl_{K'}"] \arrow[dd, bend left=60, "G"] \\ K' \arrow[d, dash, "Cl_K"] \\ K \end{tikzcd}$$

K'' is a Galois extension of K
(e.g. because $U' \subseteq \mathbb{A}_{K'}$ is invariant under the action of $Gal(K'|K)$, or because any $Gal(K'|K)$-conjugate of K'' is again an unram. abelian ext. of K' and therefore equal to K'').

$G := Gal(K''|K)$. $K''|K$ unram, $K'|K$ max. unram. ab. ext.
$K'|K$ is the max. abelian subext. of $K''|K$.
$\Rightarrow Gal(K''|K') = [G, G] \leq G$

$$\begin{tikzcd} Cl_K \arrow[r, phantom, "\cong"] \arrow[d] & Gal(K'|K) \arrow[d, "V"] \\ Cl_{K'} \arrow[r, phantom, "\cong"] & Gal(K''|K') \end{tikzcd}$$

The result follows from a theorem in group theory:

Thm Let G be any finite ~~group~~ group and $H = [G,G] \subseteq G$.
Then $V: G^{ab} \to H^{ab}$ is the trivial map.

Pf Maybe later (reinterpreting V in terms of group homology). "$\square$"