5. Class field theory

5.1. Artin reciprocity maps

Def $\left\{\begin{matrix}\text{finite}\\ \text{local}\\ \text{global}\end{matrix}\right\}$ field $K \rightsquigarrow$ topological group $C_K := \left\{\begin{matrix}\mathbb{Z} \text{ (disc. top.)}\\ K^\times \\ \mathbb{A}_K^\times / K^\times\end{matrix}\right\}$

$L|K$ finite Gal. ext. $\rightsquigarrow$ continuous action of $\mathrm{Gal}(L|K)$ on C_L
(trivial action for finite fields)

$L|K$ finite ext. $\rightsquigarrow$ cont. hom. $\mathrm{Nm}_{L|K} : C_L \longrightarrow C_K$
(mult. by $[L:K]$ for finite fields)

Thm For any K as above, there is a continuous group hom.
(Artin reciprocity map) (to be constructed later)

$$\theta_K : C_K \longrightarrow \mathrm{Gal}(K^{ab}|K)$$

satisfying a list of properties (to follow).

Prop 1 (Fin. ab. ext.) We get bijections

$$\left\{\begin{matrix}U \subseteq C_K \text{ open subgr.}\\ \text{of fin. index}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}V \subseteq \mathrm{Gal}(K^{ab}|K) \text{ open}\\ \text{(fin. index)}\end{matrix}\right\} \xleftrightarrow{\text{fund. thm. of Galois theory}} \{L|K \text{ fin. ab. ext.}\}$$

$$U = \theta_K^{-1}(V) = \boxed{\mathrm{Nm}_{L|K}(C_L)} \qquad V = \overline{\theta_K(U)} = \mathrm{Gal}(K^{ab}|L) \qquad L = (K^{ab})^V = (K^{ab})^{\theta_K(U)}$$

For any fin. ab. ext. $L|K$, we get an isom.

$$C_K / \mathrm{Nm}_{L|K}(C_L) \xrightarrow[\sim]{\theta_{L|K}} \mathrm{Gal}(L|K).$$

Cor $\mathrm{Gal}(K^{ab}|K) \left(= \varprojlim_{\substack{L|K \text{ fin.} \\ \text{ab. ext.}}} \mathrm{Gal}(L|K) \right) \cong \varprojlim_{\substack{U \subseteq C_K \\ \text{open subgr.} \\ \text{of fin. index}}} C_K/U =: \hat{C}_K$

profinite completion of C_K

Cor $\theta_K(C_K)$ is dense in $\mathrm{Gal}(K^{ab}|K)$.

Prop 2 (Functoriality)

For any fin. ext. $L|K$, we get a comm. diagram

$$\begin{tikzcd} C_L \arrow[r, "\theta_L"] \arrow[d, "\mathrm{Nm}_{L|K}"'] & \mathrm{Gal}(L^{ab}|L) \arrow[d, "\text{restriction}"] \\ C_K \arrow[r, "\theta_K"'] & \mathrm{Gal}(K^{ab}|K) \end{tikzcd}$$

$$\begin{tikzcd} K^{ab} \arrow[r, dash] \arrow[d] & L^{ab} \arrow[d, dash] \\ K \arrow[r, dash] & L \end{tikzcd}$$

Ex $K = \mathbb{F}_q$

$\mathbb{Z} \xhookrightarrow{\theta_{\mathbb{F}_q}} \hat{\mathbb{Z}} = \mathrm{Gal}(\overline{\mathbb{F}_q}|\mathbb{F}_q)$

$1 \longmapsto 1 = \varphi_q$ (Frobenius aut.)

$\{U = n\mathbb{Z} \mid n \geq 1\} \longleftrightarrow \{V = n\hat{\mathbb{Z}} \mid n \geq 1\} \longleftrightarrow \{L = \mathbb{F}_{q^n} \mid n \geq 1\}$

$\quad \parallel$

$\mathrm{Nm}_{\mathbb{F}_{q^n}|\mathbb{F}_q}(\mathbb{Z})$

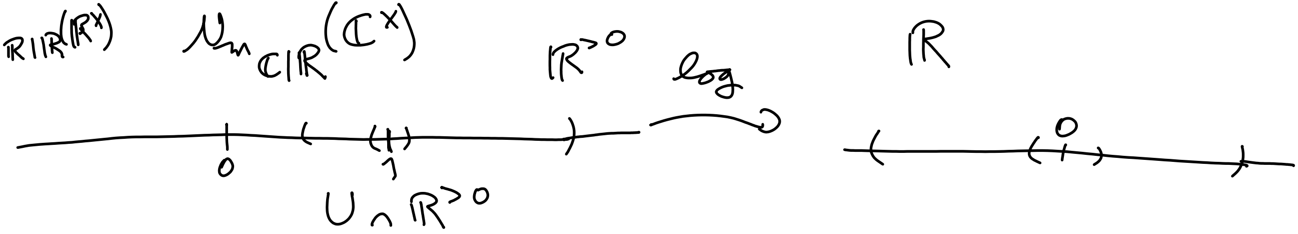
$\mathbb{Z}/n\mathbb{Z} \xrightarrow[\sim]{\theta_{\mathbb{F}_q}} \mathrm{Gal}(\mathbb{F}_{q^n}|\mathbb{F}_q)$

$1 \bmod n \longmapsto \varphi_q$

Ex $K = \mathbb{R}$

$$\mathbb{R}^\times \xrightarrow{\theta_{\mathbb{R}}} \mathrm{Gal}(\mathbb{C}|\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$$

$$\{\underset{\substack{\| \\ \mathrm{Nm}_{\mathbb{R}|\mathbb{R}}(\mathbb{R}^\times)}}{\mathbb{R}^\times}, \underset{\substack{\| \\ \mathrm{Nm}_{\mathbb{C}|\mathbb{R}}(\mathbb{C}^\times)}}{\mathbb{R}^{>0}}\} \longleftrightarrow \{\mathbb{Z}/2\mathbb{Z}, 0\} \longleftrightarrow \{\mathbb{R}, \mathbb{C}\}$$



Ex $K = \mathbb{C}$

$$\mathbb{C}^\times \xrightarrow{\theta_{\mathbb{C}}} \mathrm{Gal}(\mathbb{C}|\mathbb{C}) = 1.$$

$$\{\underset{\substack{\| \\ \mathrm{Nm}_{\mathbb{C}|\mathbb{C}}(\mathbb{C}^\times)}}{\mathbb{C}^\times}\} \longleftrightarrow \{1\} \longleftrightarrow \{\mathbb{C}\}$$

Ex K nonarch. local fields

$$C_K = K^\times = \mathcal{O}_K^\times \times \mathbb{Z}$$

$$\Rightarrow \hat{C}_K = \varprojlim_{\substack{U \subseteq K^\times \\ \text{open,} \\ \text{fin. index}}} K^\times/U = \varprojlim_{\substack{U \subseteq \mathcal{O}_K^\times \\ \text{open} \\ \text{(fin. index)}}} \mathcal{O}_K^\times/U \times \varprojlim_{\substack{U \subseteq \mathbb{Z} \\ \text{(open)} \\ \text{fin. index}}} \mathbb{Z}/U$$

$$= \widehat{\mathcal{O}_K^\times} \times \hat{\mathbb{Z}}$$

$$\underset{\substack{\uparrow \\ \boxed{\text{Lemma 5.1}}}}{=} \mathcal{O}_K^\times \times \hat{\mathbb{Z}}$$

$$\begin{tikzcd} \mathrm{CFT} \Rightarrow \mathrm{Gal}(K^{ab}|K) \cong \hat{C}_K = \mathcal{O}_K^\times \times \hat{\mathbb{Z}} \\ K^\times = \mathcal{O}_K^\times \times \mathbb{Z} . \arrow[u, hook, "\theta_K"'] \end{tikzcd}$$

Ex $K = \mathbb{Q}_p$

Local Kronecker-Weber: $\mathbb{Q}_p^{ab} = \mathbb{Q}_p(\zeta_\infty) = \bigcup_{n\geq 1} \mathbb{Q}_p(\zeta_n) = K_p \cdot \mathbb{Q}_p^{unram}$

where $K_p = \bigcup_{n\geq 1} \mathbb{Q}_p(\zeta_{p^n})$, $\mathbb{Q}_p^{unram} = \bigcup_{\substack{m\geq 1\\ p\nmid m}} \mathbb{Q}_p(\zeta_m) = \bigcup_{r\geq 1} \mathbb{Q}_p(\zeta_{p^r-1})$

a totally ramified ext.

max. (ab.) unram. ext.

$\forall m \not\equiv 0 \bmod p$: $\exists r \geq 1 : m \mid p^r-1$

$K_p \cap \mathbb{Q}_p^{unram} = \mathbb{Q}_p$

tot. ram. , unram.

$$\Rightarrow \mathrm{Gal}(\mathbb{Q}_p(\zeta_\infty)|\mathbb{Q}_p) = \mathrm{Gal}(K_p|\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p^{unram}|\mathbb{Q}_p)$$

$$= \varprojlim_{n\geq 0} (\mathbb{Z}/p^n\mathbb{Z})^\times \times \mathrm{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q)$$

$$= \mathbb{Z}_p^\times \times \hat{\mathbb{Z}} \quad ✓$$

Prop 3 (Local-finite compatibility)

Let k be a nonarch. local field with residue field $\kappa = \mathbb{F}_q$.
We get a comm. diagram

$$\begin{tikzcd}
k^\times \arrow[r, hook, "\theta_k"] \arrow[d, two heads, "v_k"'] & \mathrm{Gal}(k^{ab}|k) \arrow[r, equal] \arrow[d, two heads, "\text{reduction mod } \mathfrak{p}_{k^{ab}}"'] & D \arrow[r, equal] \arrow[d, two heads] & \mathrm{Gal}(k^{ab}|k) \arrow[d, two heads, "\text{restriction}"] \\
\mathbb{Z} \arrow[r, hook, "\theta_\kappa"'] & \mathrm{Gal}(\kappa^{ab}|\kappa) \arrow[r, equal] & D/I \arrow[r, equal] & \mathrm{Gal}(k^{unram}|k)
\end{tikzcd}$$

Cor

$$\begin{array}{ccccc} \mathrm{Gal}(k^{ab}|k) & = & \hat{C}_k & = & \mathcal{O}_k^\times \times \hat{\mathbb{Z}} \\ \cup\!| & & & & \cup\!| \\ \rightsquigarrow I(k^{ab}|k) & = & & & \mathcal{O}_K^\times \end{array}$$

Prop 4 (Global-local compatibility)

Let K be a global field, and v be a place of K.

$$\begin{tikzcd} \mathbb{A}_K^\times / K^\times \arrow[r, "\theta_K"] & \mathrm{Gal}(K^{ab}|K) \\ K_v^\times \arrow[u, hook, "\text{embedding in v-coord.}"] \arrow[r, "\theta_{K_v}"'] & \mathrm{Gal}(K_v^{ab}|K_v) = D(w|v) \arrow[u, hook] \end{tikzcd}$$

for any ext. w of v from K to K^{ab} (well-def. subgroup of $\mathrm{Gal}(K^{ab}|K)$ (independent of choice of w) because all decomposition groups are conjugate and therefore identical in abelian extensions)

Some ideas for final papers

1. Witt vectors

$$\mathbb{Z}_p \rightsquigarrow \mathbb{F}_p$$

$$\mathbb{Z}_p \leftsquigarrow \mathbb{F}_p \qquad \qquad ~~\{0, 1, \ldots, p-1\}~~$$

$$\mathbb{Z}_q \leftsquigarrow \mathbb{F}_q \qquad \qquad \{0\} \cup \mu_{p-1}$$

2. Complex multiplication

What is the max. ab. ext. of a given imaginary quadratic number field? (Has to do with elliptic curves!)

3. Tropical geometry

Newton polygons tell you what the valuations of the roots of a polynomial $\in K[X]$ are.

More generally, what are the valuations of the points on a variety V?

$\rightsquigarrow$ Fundamental theorem of tropical geometry

Transverse intersection theorem

4. Cubic and higher reciprocity laws

Know quadratic reciprocity.

How to generalize? (. . . .)

5. Hasse-Minkowski theorem

$\{f(x_1, \ldots, x_n) = 0\}$ for hom. degree 2 pol. f satisfies the Hasse principle.

6. Nonarch local analysis

Haar measure on any local field