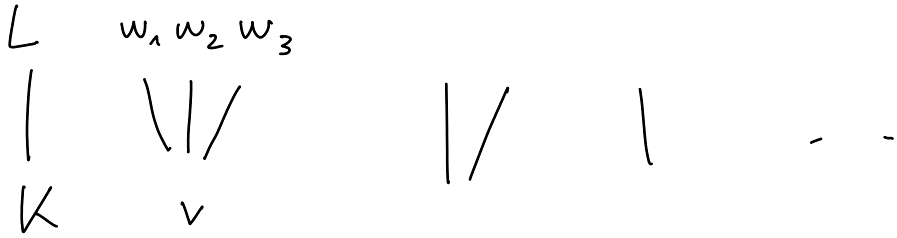
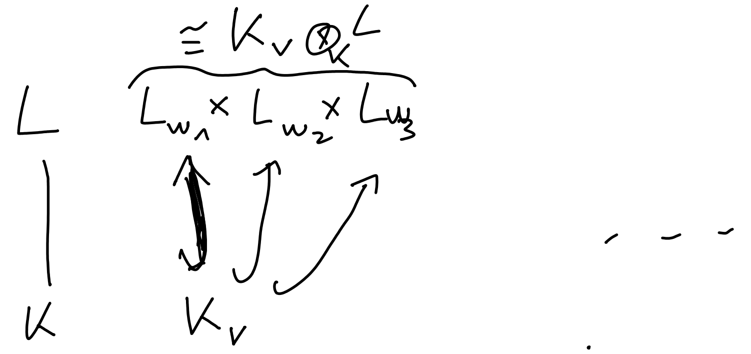
4.3. Adèles in extension land

Let $L|K$ be a separable ext. of global fields.





$\Rightarrow$ **Thm 4.3.1** $\quad \mathbb{A}_L \overset{\text{as rings}}{\cong} \mathbb{A}_K \otimes_K L$

$$\mathbb{A}_L \overset{\text{as top. groups}}{\cong} \underbrace{\mathbb{A}_K \times \dots \times \mathbb{A}_K}_{[L:K]}$$

Rmk $\mathbb{A}_K \subseteq \mathbb{A}_L$ is cont.

Rmk If $\sigma \in \mathrm{Gal}(L|K)$, we get an automorphism σ of $\underset{\mathbb{A}_L}{\underset{\parallel}{\mathbb{A}_K \otimes_K L}}$: $x \otimes y \longmapsto x \otimes \sigma(y)$.

Explicitly, $\sigma(x_w)_w = (\sigma\, x_{w\circ\sigma})_w$

$$= (\sigma\, x_{\sigma^{-1}w})_w.$$

Def Trace $\mathrm{Tr}_{L|K}: \mathbb{A}_L \longrightarrow \mathbb{A}_K$

$$(x_w)_w \longmapsto \Big(\sum_{w|v} \mathrm{Tr}_{L_w|K_v}(x_w)\Big)_v \quad \Big(= \sum_{\sigma \in \mathrm{Gal}} \sigma x \text{ if } L|K \text{ Galois}\Big)$$

Norm $\mathrm{Nm}_{L|K}: \mathbb{A}_L \longrightarrow \mathbb{A}_K$

$$(x_w)_w \longmapsto \Big(\prod_{w|v} \mathrm{Nm}_{L_w|K_v}(x_w)\Big)_v \Big(= \prod_{\sigma} \sigma x \text{ if } L|K \text{ Galois}\Big)$$

4.4. Approximation theorems

Let K be a global field.

Weak approximation theorem

Let S be a finite set of places of K. Then, the map

$$K \longrightarrow \prod_{v \in S} K_v \quad \text{has dense image.}$$

Strong approximation theorem (away from S)

Let S be a nonempty set of places of K. Let

$$\mathbb{A}_K^S := \{ (x_v)_{v \notin S} \in \prod_{v \notin S} K_v \mid x_v \in \mathcal{O}_v \text{ for almost all } v \} = \prod_{v \notin S}{}' K_v.$$

(restricted product)

Then, the map $K \hookrightarrow \mathbb{A}_K^S$ has dense image.

Note It suffices to prove this for every 1-element set S.

Ex $K = \mathbb{Q}$, $S = \{\infty\}$.

Open base of $\mathbb{A}_K^S$: $U = \prod_p U_p$, $y_p + p^{e_p}\mathbb{Z}_p \subseteq U_p \subseteq \mathbb{Q}_p$ open $\forall p$

$(y_p \in \mathbb{Q}_p,\ e_p \in \mathbb{Z})$

$U_p = \mathbb{Z}_p$ for a.a. p

Goal: $\exists x \in \mathbb{Q}$: $x \in y_p + p^{e_p}\mathbb{Z}_p$ for fin. many p

$x \in \mathbb{Z}_p$ for all other p.

Multiplying by powers of p, we can make $y_p \in \mathbb{Z}_p$, $e_p \geq 0$.

Use the Chinese remainder theorem.

Ex $K = \mathbb{Q}$, $S = \{2\}$.

Open base of $\mathbb{A}_K^S$: $U = \prod_{p \neq 2} U_p \times U_\infty$, $y_p + p^{e_p}\mathbb{Z}_p \subseteq U_p \subseteq \mathbb{Q}_p \ \forall p \neq 2$

$U_p = \mathbb{Z}_p$ for a.a. $p \neq 2$

$(r, s) \subseteq U_\infty \subseteq \mathbb{R}$ open

Goal: $\exists x \in \mathbb{Q}$: $x \in y_p + p^{e_p}\mathbb{Z}_p$ for fin. many $p \neq 2$.

$x \in \mathbb{Z}_p$ for all other $p \neq 2$

$x \in (r, s)$.

Multiplying by powers of $p \neq 2$, we can make $y_p \in \mathbb{Z}_p$, $e_p \geq 0$. $\forall p \neq 2$.

Multiplying by a large power of 2, we can make $s - r > \prod_{p \neq 2} p^{e_p}$.

Use the Chinese remainder theorem.

Pf See Cassels-Fröhlich (Alg. Number Theory): Chapter II. 15. $\square$

More generally, one studies the following properties:

Def A variety V defined over K <u>satisfies weak approximation at S</u> if $V(K) \longrightarrow V(\prod_{v \in S} K_v)$ has dense image.

Def Say K is a number field. A variety V defined over $\mathcal{O}_K$ <u>satisfies strong approximation away from S</u> if $V(K) \hookrightarrow V(\mathbb{A}_K^S)$ has dense image.

Rmk We showed that the affine line $\mathbb{A}^1$ satisfies strong approximation.

4.5. Cocompactness

Thm 4.5.1 $\mathbb{A}_K/K$ is compact for any global field K.

Rmk By Thm 4.3.1., it suffices to show this for $K=\mathbb{Q}, \mathbb{F}_p(T)$.

Lemma Let $\mathcal{O}_K$ be the integral closure of $\left\{\begin{matrix}\mathbb{Z}\\ \mathbb{F}_p[T]\end{matrix}\right\}$ in K.

Then,
$$\mathbb{A}_K/K \cong \left(\prod_{v\nmid\infty} \mathcal{O}_v \times \prod_{v\mid\infty} K_v\right)\Big/\mathcal{O}_K.$$

Pf "$\mapsto$" strong approximation

"$\leftarrow$" $\{x\in K \mid x\in\mathcal{O}_v\ \forall v\nmid\infty\} = \mathcal{O}_K$. $\square$

Pf of 4.5.1 for $K=\mathbb{Q}$

$$\mathbb{A}_\mathbb{Q}/\mathbb{Q} \cong \left(\prod_p \mathbb{Z}_p \times \mathbb{R}\right)\Big/\mathbb{Z}$$
$$\uparrow\uparrow$$
$$\prod_p \mathbb{Z}_p \times [0,1] \text{ compact}$$
$\square$

Pf of 4.5.1 for any number field K

$$\mathbb{A}_K/K \cong \left(\prod_{\mathfrak{p}} \mathcal{O}_\mathfrak{p} \times (K\otimes_\mathbb{Q}\mathbb{R})\right)\Big/\mathcal{O}_K$$
$$\uparrow\uparrow$$
$$\prod_{\mathfrak{p}} \mathcal{O}_\mathfrak{p} \times ([0,1]\cdot\omega_1 + \dots + [0,1]\cdot\omega_n) \text{ compact,}$$

where $\omega_1,\dots,\omega_n$ is an integral basis of K. $\square$

Pf of 4.5.1 for $K=\mathbb{F}_p(T)$

$$\mathbb{A}_{\mathbb{F}_p(T)}/\mathbb{F}_p(T) \cong \left(\prod_f \mathcal{O}_f \times \mathbb{F}_p((\tfrac{1}{T}))\right)\Big/\mathbb{F}_p[T]$$
$$\uparrow\uparrow$$
$$\prod_f \mathcal{O}_f \times \{f\in\mathbb{F}_p((\tfrac{1}{T})) \mid v_\infty(f)\geq 0\} \text{ compact}$$
$\square$

4.6. Idèles

Group of idèles $\mathbb{A}_K^\times$.

Trouble $\mathbb{A}_K^\times \longrightarrow \mathbb{A}_K^\times$, $x \longmapsto x^{-1}$ is not continuous w.r.t. the subspace topology! $\rightsquigarrow$ Using the subspace top. doesn't yield a top. group!

Pf $U := \left(\prod_{v \text{ nonarch}} \mathcal{O}_v \times \prod_{v \text{ arch.}} K_v\right) \cap \mathbb{A}_K^\times$ open w.r.t. subsp. top.

$$= \{(x_v)_v \in \mathbb{A}_K^\times \mid v(x_v) \geq 0 \ \forall v \text{ nonarch}\}.$$

$$U^{-1} = \{(x_v)_v \in \mathbb{A}_K^\times \mid v(x_v) \leq 0 \ \forall v \text{ nonarch}\}.$$

doesn't contain any nonempty open subset of $\mathbb{A}_K^\times$.

$$\left(\prod U_v, \quad U_v = \mathcal{O}_v \text{ for a.a. } v\right). \qquad \square$$

Fise $\mathbb{A}_K^\times \cong \{(x,y) \in \mathbb{A}_K \times \mathbb{A}_K \mid xy = 1\}$ as groups
$x \longmapsto (x, x^{-1})$

Use the subspace top. on the RHS $\subseteq \mathbb{A}_K \times \mathbb{A}_K$.

$\rightsquigarrow \mathbb{A}_K^\times$ is automatically a topological group!

Rmk Open base for top. on $\mathbb{A}_K^\times$:

$\prod_v U_v$, where $U_v \subseteq K_v^\times$ open $\forall v$,
$U_v = \mathcal{O}_v^\times$ for a.a. (nonarch.) v.

Rmk $K^\times \subseteq \mathbb{A}_K^\times$ is discrete and closed.

Def The <u>idèle class group</u> of K is $\mathbb{A}_K^\times / K^\times$.

We have a <u>content map</u> $c : \mathbb{A}_K^\times \longrightarrow \mathbb{R}^{>0}$

$$(x_v)_v \longmapsto \prod_v |x_v|_v$$

fin. prod. because $x_v \in \mathcal{O}_v^\times$ and therefore $|x_v|_v = 1$ for a.a. v

Rmk $c(\mathbb{A}_K^\times) = \begin{cases} \mathbb{R}^{>0}, & K \text{ number field} \\ q^{\mathbb{Z}} \ (?), & K \text{ function field with residue field } \mathbb{F}_q. \end{cases}$

is an infinite subset of $\mathbb{R}^{>0}$.

Def $J_K^1 := \ker(c) = \{ (x_v)_v \in \mathbb{A}_K^\times \mid \prod_v |x_v|_v = 1 \}$.

Rmk Product formula: $K^\times \subseteq J_K^\times$.

$\Rightarrow \mathbb{A}_K^\times / K^\times$ is not compact (image of $\mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{R}^{>0}$ isn't compact)

Thm $J_K^1 / K^\times$ is compact.

Pf See Cassels-Fröhlich: Chapter II.16. $\square$

Ex $K = \mathbb{Q}$.

$$J_{\mathbb{Q}}^1 / \mathbb{Q}^\times \cong \prod_p \mathbb{Z}_p^\times = \hat{\mathbb{Z}}^\times.$$

$$[(x_2, x_3, \ldots, x_\infty)] \longmapsto (x_2, x_3, \ldots)$$

$|x_2|_2 |x_3|_3 \cdots |x_\infty|_\infty = 1$

multiply by appropriate power of p to make $x_p \in \mathbb{Z}_p^\times$ ($\Leftrightarrow |x_p|_p = 1$)

multiply by ± 1 to make $x_\infty > 0$

$\Rightarrow x_\infty = 1$

Thm Let K be a number field.

$$\Rightarrow \mathbb{A}_K^\times \Big/ K^\times \cdot \left(\prod_{v\nmid\infty} \mathcal{O}_v^\times \times \prod_{v\mid\infty} K_v^\times\right) \cong \mathcal{Cl}_K$$

(the ideal class group)

Pf
$$LHS \cong \prod_{\mathfrak{p}}{}' \left(K_{\mathfrak{p}}^\times / \mathcal{O}_{\mathfrak{p}}^\times\right) \Big/ K^\times \cong \left(\prod_{\mathfrak{p}}{}' \mathbb{Z}\right) \Big/ K^\times$$

$$[(x_{\mathfrak{p}})_{\mathfrak{p}}] \longmapsto [(v_{\mathfrak{p}}(x_{\mathfrak{p}}))_{\mathfrak{p}}]$$

$$\cong \left(\prod_{\mathfrak{p}}{}' \mathfrak{p}^{\mathbb{Z}}\right) \Big/ K^\times \cong (\text{frac. ideals of } K) / K^\times.$$

$\square$

Cor We get an exact sequence

$$1 \longrightarrow \left(\prod_{v\nmid\infty} \mathcal{O}_v^\times \times \prod_{v\mid\infty} K_v^\times\right) \Big/ \mathcal{O}_K^\times \longrightarrow \mathbb{A}_K^\times / K^\times \longrightarrow \mathcal{Cl}_K \longrightarrow 1.$$