Last week

Gal$(L|K)$ compact

compact $\Rightarrow$ ~~sequentially compact~~

(correct for countable products of compact spaces)

Wyatt: Example where Gal$(L|K)$ is not sequentially compact

$$K = \mathbb{R}(T)$$

$$L = K(\{\sqrt{T-\lambda} \mid \lambda \in \mathbb{R}\}).$$

$$\Rightarrow \mathrm{Gal}(L|K) = \underbrace{\prod_{\lambda \in \mathbb{R}}}_{\text{not sequentially compact}} \mathbb{Z}/2\mathbb{Z} \text{ with prod. topology}$$

For any finite, local, global field K, Gal$(K^{sep}|K)$ is sequentially compact, because there are only countably many finite Galois extensions of K.

Preview

How to tell whether $K \subseteq \mathbb{Q}(\zeta_\infty)$?

Surprise:

Kronecker-Weber Theorem

$\mathbb{Q}(\zeta_\infty)$ is the maximal abelian extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$.

Equivalently: A fin. field ext. $K \mid \mathbb{Q}$ is abelian if and only if $K \subseteq \mathbb{Q}(\zeta_n)$ for some $n \geq 1$.
The smallest such n (= gcd of all such n) is called the conductor of K.

Ex $K = \mathbb{Q}(\sqrt{a})$ is an abelian ext.

Its conductor is $|\underbrace{\mathrm{disc}}_{\text{discriminant of } K}(K)|$.

Local Kronecker-Weber Theorem

$\mathbb{Q}_p(\zeta_\infty) = \bigcup_{n \geq 1} \mathbb{Q}_p(\zeta_n)$ is the max. abelian ext. of $\mathbb{Q}_p$.

> Slightly dangerous notation:
> The primitive n-th roots of unity might not be Galois conjugate over $\mathbb{Q}_p$.
> But they all generate the same field ext. of $\mathbb{Q}_p$.

Questions What are the max. ab. ext. of other number fields / local fields K? What is $\mathrm{Gal}(K^{ab} \mid K)$? How to compute the conductor of an abelian extension?

1.9. Normalised absolute values

<u>Def</u> Let K be a local field.

$$|x|_K = q_K^{-v_K(x)}$$ if K is nonarch. with res. field $\mathbb{F}_{q_K}$, normalised disc. val. v_K.

$|x|_{\mathbb{R}} = |x|$, the usual abs. value if $K = \mathbb{R}$

$|x|_{\mathbb{C}} = |x|^2 = |x \cdot \bar{x}|$ if $K = \mathbb{C}$

Doesn't satisfy the triangle inequality.

<u>Lemma 1.6.1</u> For any (fin) ext. $L|K$ of local fields,

$$|x|_L = |\mathrm{Nm}_{L|K}(x)|_K \qquad \forall x \in L.$$

<u>Pf</u> <u>L, K nonarch.</u>:

$$q_L = q_K^f, \quad v_L(x) = e \cdot v_K(x) = e \cdot \frac{1}{n} \cdot v_K(\mathrm{Nm}_{L|K}(x)),$$

$n = e \cdot f$.

<u>$L = \mathbb{C}, K = \mathbb{R}$</u> clear. $\square$

4. Global fields

<u>Def</u> A <u>global field</u> K is

a) a fin. ext. of $\mathbb{Q}$ (<u>number field</u>)

b) a fin. (separable) ext. of $\mathbb{F}_p(T)$ ((global) <u>function field</u>).

4.1. Places

For any disc. val. v on K, we get a local field $\hat{K}_v$ with ring of integers $\hat{\mathcal{O}}_v$. There's a natural embedding $K \hookrightarrow \hat{K}_v$.

Change of notation: $K_v := \hat{K}_v$, $\mathcal{O}_v := \hat{\mathcal{O}}_v$.

If K is a number field, we also have real embeddings $K \hookrightarrow \mathbb{R}$, pairs of complex embeddings $K \hookrightarrow \mathbb{C}$.

<u>Def</u> A <u>place</u> v of K is

- a (norm.) disc. val. v, leading to an emb. $K \hookrightarrow K_v$ } finite (nonarch.) place
- an embedding $K \hookrightarrow \mathbb{R}$ $(K_v := \mathbb{R})$
- a pair of complex conj. emb. $K \hookrightarrow \mathbb{C}$ $(K_v := \mathbb{C})$ } infinite (arch.) place.

<u>Rmk</u> The places are the equivalence classes of multiplicative valuations ~~on~~ K (cf. Neukirch, II.3, III.1)

<u>Eg</u> The places of $\mathbb{Q}$ are the prime numbers $v = p$ and $v = \infty$.
(the real embedding)

<u>Def</u> If $L|K$ is an ext of global fields, v is a place of K, w is a place of L, we write $w|v$ if $K \hookrightarrow K_v$ "is" the restriction of $L \hookrightarrow L_w$ to K.

The cases are:

- $v = v_{\mathfrak{q}}$, $w = v_{\mathfrak{P}}$, $\mathfrak{P} | \mathfrak{q}$
  $\mathfrak{q} \subseteq \mathcal{O}_K$, $\mathfrak{P} \subseteq \mathcal{O}_L$

- $v : K \hookrightarrow \mathbb{R}, \mathbb{C}$, $w : L \hookrightarrow \mathbb{R}, \mathbb{C}$, $w|_K = v$

<u>Eg</u> The places of $\mathbb{Q}(\sqrt{2})$ are the primes and ∞_1, ∞_2
real emb.

$\infty_1, \infty_2 | \infty$.

<u>Lemma</u> For any fin. separable ext. $L|K$ of global fields and any place v of K,

$$\prod_{w|v} |x|_w = |N_{L|K}(x)|_v.$$

<u>Pf</u> $L|K$ is separable $\Rightarrow L \otimes_K K_v \cong \prod_{w|v} L_w$.

$$\prod_{w|v} |x|_w \overset{\boxed{\text{Lemma 1.6.1}}}{=} \prod_{w|v} |N_{L_w|K_v}(x)|_v = \left| \prod_{w|v} N_{L_w|K_v}(x) \right|_v$$

$$= \left| N_{L \otimes_K K_v | K_v}(x) \right|_v = |N_{L|K}(x)|_v. \quad \square$$

<u>Thm (Product Formula)</u> Let K be a global field.

$$\Rightarrow \prod_v |x|_v = 1 \qquad \forall x \in K^\times.$$

Pf for $K = \mathbb{Q}$

$$x = \pm \prod_p p^{a_p}. \quad \Rightarrow \quad |x|_p = p^{-a_p} \qquad \forall p$$

$$|x|_\infty = \prod_p p^{a_p}$$

$$\prod_v |x|_v = 1.$$

$\square$

Pf for $K = \mathbb{F}_q(T)$

$$x = \lambda \cdot \prod_{\substack{f(T) \text{ monic} \\ \text{irred.}}} f(T)^{a_f} \qquad (\lambda \in \mathbb{F}_q^\times)$$

$\Rightarrow |x|_f = q^{-\deg(f) \cdot a_f}$ (res. field $\mathbb{F}_q[T]/f(T)$ has size $q^{\deg(f)}$).

$|x|_\infty = q^{\deg(x)} = q^{\sum_f \deg(f) \cdot a_f}$ (res. field $\mathbb{F}_q[\frac{1}{T}]/(\frac{1}{T}) = \mathbb{F}_q$).

$\square$

Pf for general K

Say K is a fin. ext. of $\mathbb{Q}$.

$$\Rightarrow \prod_{w \text{ pl. of } K} |x|_w = \prod_{v \text{ pl. of } \mathbb{Q}} \prod_{w|v} |x|_w \underset{\text{Lemma}}{=} \prod_v |N_{K|\mathbb{Q}}(x)|_v = 1.$$

Same for fin. ext. of $\mathbb{F}_q(T)$.

$\square$

4.2. Adèles

<u>Motivation</u> Let $f(x_1,\dots,x_n) \in \mathbb{Q}[x_1,\dots,x_n]$.

$\rightsquigarrow V = \{(x_1,\dots,x_n) : f(x_1,\dots,x_n) = 0\}$.

Assume $V(\mathbb{Q}) \neq \emptyset$.

$\Rightarrow V(\mathbb{Q}_p) \neq \emptyset \ \forall p \ , \ V(\mathbb{R}) \neq \emptyset$

$\Leftrightarrow V(\prod_p \mathbb{Q}_p \times \mathbb{R}) \neq \emptyset$.

Note that any $x \in \mathbb{Q}$ lies in $\mathbb{Z}_p$ for all but finitely many p (those not dividing the denominator of x).

<u>Def</u> The <u>adèle ring</u> $\mathbb{A}_K$ is the ring of tuples $(x_v)_v \in \prod_v K_v$ such that $x_v \in \mathcal{O}_v$ for all but finitely many nonarch. places v.

<u>Rmk</u> $K \subset \mathbb{A}_K$.
$x \mapsto (x)_v$.
In part., if $V(K) \neq \emptyset$, then $V(\mathbb{A}_K) \neq \emptyset$.

<u>Def</u> Define a topology on $\mathbb{A}_K$ with open base consisting of sets of the form $\prod_v U_v$, where all $U_v \subseteq K_v$ are open, and $U_v = \mathcal{O}_v$ for all but finitely many nonarch. places v.

<u>Rmk</u> $\mathbb{A}_K$ is a topological ring:
$+ : \mathbb{A}_K \times \mathbb{A}_K \to \mathbb{A}_K \ , \ \times : \mathbb{A}_K \times \mathbb{A}_K \to \mathbb{A}_K$
are continuous.

Remark $K \subseteq \mathbb{A}_K$ is discrete.

Pf. It suffices to prove that for any $x \in K$, there is an open set $U \subseteq \mathbb{A}_K$ such that $K \cap U = \{x\}$.

W.l.o.g. $x = 0$.

Fix a nonempty finite set S of places containing all arch. places.

Take $$U = \prod_{v \notin S} \underbrace{\{x \in K_v \mid |x|_v \leq 1\}}_{= \mathcal{O}_v} \times \prod_{v \in S} \underbrace{\{x \in K_v \mid |x|_v < 1\}}_{\text{open}}.$$

By the product formula, U contains no element of K other than 0. $\square$

Remark $K \subseteq \mathbb{A}_K$ is closed.

Remark $\mathbb{A}_K / K$ is compact.