Math 223a: Algebraic Number Theory

Fall 2019

Homework #8

due Tuesday, November 5 at noon

Problem 1. Let $L|K$ be a finite field extension with Galois group $\mathrm{Gal}(L|K) \cong \mathbb{Z}/n\mathbb{Z}$ generated by σ.

a) Let $a \in L$. Show that there exists some $b \in L$ such that $a = b - \sigma(b)$ if and only if $\mathrm{Tr}_{L|K}(a) = 0$. (*Additive Hilbert 90*)

b) Assume K is a local field. Show that there exists some integer $n \geqslant 0$ such that the following holds for all $a \in \mathfrak{p}_K^n$: There exists some $b \in \mathcal{O}_L$ such that $a = b - \sigma(b)$ if and only if $\mathrm{Tr}_{L|K}(a) = 0$. (*Integral additive Hilbert 90*)

Problem 2. Let $(a,b) = \left(\frac{a,b}{(2)}\right)_2$ be the Hilbert symbol for $K = \mathbb{Q}_2$ and $n = 2$. Show that

$$(2^s \cdot a, 2^t \cdot b) = (-1)^{s \cdot \frac{b^2-1}{8} + t \cdot \frac{a^2-1}{8} + \frac{a-1}{2} \cdot \frac{b-1}{2}}$$

for all $a, b \in \mathbb{Z}_2^\times$ and $s, t \in \mathbb{Z}$.

Problem 3.

a) Is there a field extension $L|\mathbb{Q}$ with a basis (α, β, γ) such that $N_{L|\mathbb{Q}}(\alpha a + \beta b + \gamma c) = a^3 + 7b^3 + 49c^3 - 21abc$ for all $a, b, c \in \mathbb{Q}$?

b) Is there a field extension $L|\mathbb{Q}$ with a basis (α, β, γ) such that $N_{L|\mathbb{Q}}(\alpha a + \beta b + \gamma c) = a^3 + 7b^3 + 49c^3 - 21abc + a^2c$ for all $a, b, c \in \mathbb{Q}$?

Problem 4. Show that an element x of $\mathbb{Q}_3^\times$ can be written as

$$x = a^4 - 4a^2b^2 + 4b^4 - 6a^2c^2 - 12b^2c^2 + 9c^4 + 48abcd - 12a^2d^2 - 24b^2d^2 - 36c^2d^2 + 36d^4$$

with $a, b, c, d \in \mathbb{Q}_3$ if and only if $x \in \mathbb{Q}_3^{\times 2}$. (You may use a computer algebra system.)