Math 223a: Algebraic Number Theory

Fall 2019

Homework #6

due Tuesday, October 22 at noon

Problem 1. Let K be a local field. Show that the field K^{ur} is not complete with respect to the discrete valuation v_K.

Problem 2. Let K be a local field with residue field $\mathbb{F}_q$. Let $f(X)$ be an Eisenstein polynomial of degree $q-1$ with constant coefficient π_K and let $\alpha \in \overline{K}$ be a root of $f(X)$. Consider the field $L = K(\alpha)$. Show that $\mathrm{Nm}_{L|K} L^\times$ is $U_K^{(1)} \cdot \pi_K^{\mathbb{Z}}$, the set of elements x of $K^\times$ such that $x/\pi_K^{v_K(x)} \equiv 1 \mod \mathfrak{p}_K$.

Definition. A *cyclic extension* $L|K$ is a Galois extension with cyclic Galois group. A *dihedral extension* $L|K$ is a Galois extension with dihedral Galois group (group of order $2n$ generated by τ and σ subject to the relations $\tau^n = 1$, $\sigma^2 = 1$, $\sigma\tau\sigma = \tau^{-1}$).

Problem 3. Let K be a local field with residue field $\mathbb{F}_q$.

a) For which integers $n \geqslant 1$ does K have a totally and tamely ramified cyclic extension of degree n?

b) For which integers $n \geqslant 1$ does K have a tamely ramified dihedral extension of degree $2n$?

Problem 4. Let K be a local field with residue field $\mathbb{F}_q$ of odd order q. Show that there is a Galois extension $L|K$ with $\mathrm{Gal}(L|K) \cong \mathbb{Z}/4\mathbb{Z}$ and $I(L|K) \cong \mathbb{Z}/2\mathbb{Z}$.

Problem 5. Let $K|\mathbb{Q}_p$ be a Galois extension with Galois group $\mathbb{Z}_p$. Show that $I_0(K|\mathbb{Q}_p) = I_1(K|\mathbb{Q}_p)$.