Math 223a: Algebraic Number Theory

Fall 2019

Homework #5

due Thursday, October 10 at noon

Problem 1. For any integer $n \geqslant 2$, denote by $\Phi_n(X) \in \mathbb{Z}[X]$ the n-th cyclotomic polynomial. Let p be a prime number. For which numbers $n \geqslant 1$ does there exists an integer d such that $\Phi_n(X+d)$ is an Eisenstein polynomial in $\mathbb{Q}_p[X]$?

Problem 2. Let $K \subseteq \mathbb{Q}(\zeta_\infty)$ be a finite field extension of $\mathbb{Q}$. For any prime p, let $k = k_p$ be the smallest nonnegative integer such that $I^k(\mathfrak{p}|p) = 1$, where $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_K$ dividing p. (In particular, $k_p = 0$ if and only if p is unramified.) Show that the conductor of K is $\prod_p p^{k_p}$.

Problem 3. Let $p \geqslant 3$ be any odd prime and let $n \geqslant 4$. Show that $\mathbb{Q}_p$ doesn't have a Galois extension with Galois group S_n.

Problem 4. Let $n \geqslant 1$ and let K be a local field of characteristic 0. Show that K has only finitely many field extensions of degree n.