Math 223a: Algebraic Number Theory

Fall 2019

Homework #10

due Tuesday, November 19 at noon

Problem 1. Let G be a finite group and let A be a (co-)induced G-module. Let $I \subseteq \mathbb{Z}[G]$ be the augmentation ideal and let $N : A \to A$ be the map sending a to $\sum_{g \in G} ga$. Show that the image of N is A^G and its kernel is IA.

Problem 2. Let G be a finite group. Show that $H^1(G, \mathbb{Z}) = 0$ and $H^2(G, \mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$.

Problem 3.
a) Let $G = S_3$ act on the group $A = \mathbb{Z}^3$ by permuting the coordinates. Compute the cohomology groups $H^n(G, A)$ for $n \geqslant 0$.

b) Let $G = S_3$ act on the group $B = \mathbb{Z}^2$ as follows: $\pi.(x, y) = (x, y)$ if $\pi \in S_3$ is an even permutation and $\pi.(x, y) = (y, x)$ otherwise. Compute the cohomology groups $H^n(G, B)$ for $n \geqslant 0$.

Problem 4. Prove the normal basis theorem for finite fields: Any extension $\mathbb{F}_{q^n}|\mathbb{F}_q$ of finite fields has a basis of the form $\{g(x) \mid g \in \mathrm{Gal}(\mathbb{F}_{q^n}|\mathbb{F}_q)\}$ with $x \in \mathbb{F}_{q^n}$. (Hint: Make $\mathbb{F}_{q^n}$ a $\mathbb{Z}[X]$-module and apply the fundamental theorem of finitely generated modules over principal ideal domains.)