Math 223a: Algebraic Number Theory

Fall 2019

Some ideas for final papers

Here are some ideas for the 5–10 page final papers. You are of course more than welcome to come up with your own topics!

1. *Witt vectors:* They provide a way to construct the ring $\mathbb{Z}_p$ "directly" from its residue field $\mathbb{F}_p$, and its unramified degree n extension $\mathbb{Z}_{p^n}$ from the residue field $\mathbb{F}_{p^n}$. Witt managed to use this to explicitly describe cyclic extensions of K of degree p^n when $\mathrm{char}(K) = p$. (The case $n = 1$ is called Artin–Schreier theory.) See for example [Bos18, Sections 4.8–4.10].

2. *Complex multiplication:* We have very explicitly constructed the maximal abelian extension K^{ab} of a field K when K is a local field or $K = \mathbb{Q}$. The case of general number fields is more difficult. The theory of complex multiplication provides a way to construct K^{ab} when K is a quadratic imaginary number fields. This involves looking at particular elliptic curves associated to K. (*Kronecker's Jugendtraum*) See for example [Cox13] or [67, Chapter XIII] or [ST15, Chapter 6] or [Sil94, Chapter II].

3. *Tropical geometry:* This can be viewed as a generalization of Newton polygons. Say we have polynomials $f_1, \dots, f_k \in \mathcal{O}_K[X_1, \dots, X_n]$. Can we determine the set of possible valuation tuples $(v_K(\alpha_1), \dots, v_K(\alpha_n)) \in \mathbb{R}^n$ for solutions $\alpha = (\alpha_1, \dots, \alpha_n) \in \overline{K}^n$ to $f_1(\alpha) = \cdots = f_k(\alpha) = 0$? (Newton polygons completely answered that question for $k = n = 1$.) You could for example look at [RST05] or [Mik04].

4. *Database of local fields:* Jones and Roberts compiled a list of all field extensions $K|\mathbb{Q}_p$ of degree $\leqslant 10$ with $p \leqslant \dots$. How did they algorithmically tell apart two local fields? How did they know when they had found all of them? See [JR06].

5. *Algorithmic number theory:* There are lots of interesting algorithms in number theory. For example: How do you efficiently factor a polynomial $f(X) \in \mathbb{Q}[X]$? See for example [Coh93] or [Coh00].

6. *Cubic and higher reciprocity laws:* This is a generalization of the quadratic reciprocity law that can be tackled using Hilbert symbols coming from the Artin reciprocity map. See for example [Cox13] or [Lem00].

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