UNIVERSITÄT BIELEFELD

UNIVERSITÄT PADERBORN

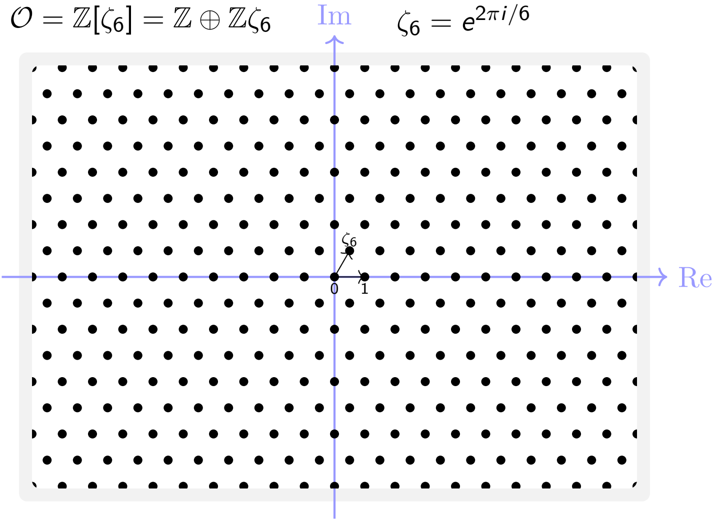
Project A2
Algebraic and arithmetic aspects of aperiodicity (and alliterations)

Fabian Gundlach

February 19, 2025

TRR 358

$\mathcal{O} = \mathbb{Z}[\zeta_6] = \mathbb{Z} \oplus \mathbb{Z}\zeta_6$ $\zeta_6 = e^{2\pi i/6}$



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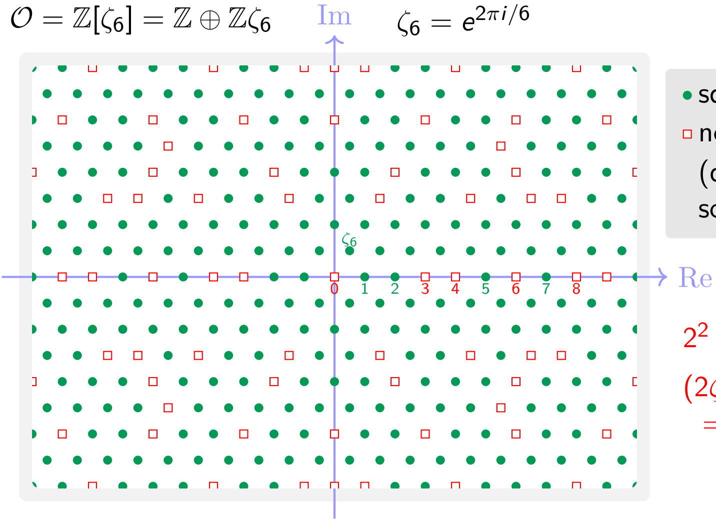

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- ▫ not squarefree (divisible by some $\mathfrak{p}^2$)

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$(2\zeta_6 + 1)^2 = (\sqrt{-3})^2 \mid 3$

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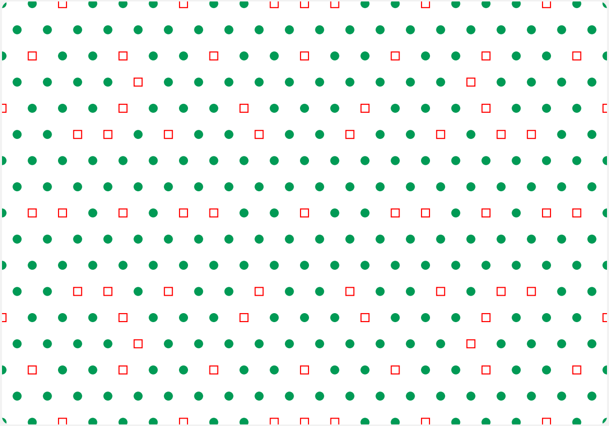

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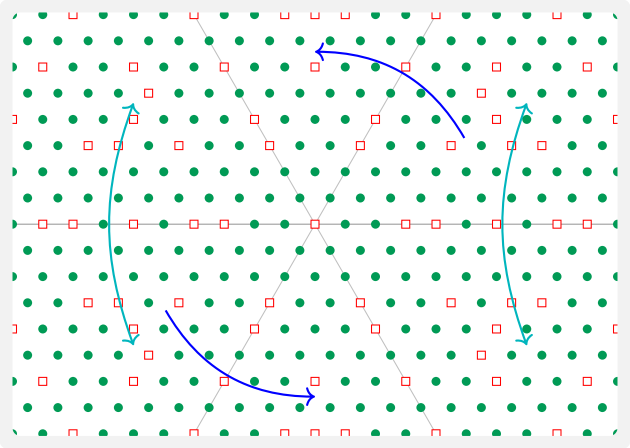

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Results

Theorem (Gundlach–Klüners, 2024; generalizing Baake–Bustos–Nickel, 2023)

$\mathcal{O}$ *ring of integers of a number field*

All (affine) $\mathbb{Z}$*-linear bijections* $f : \mathcal{O} \to \mathcal{O}$
preserving $\{a \in \mathcal{O} \text{ squarefree}\}$
are of the form $a \mapsto u \cdot \tau(a)$
with $u \in \mathcal{O}^\times$ *and* $\tau \in \mathrm{Aut}_{\mathrm{ring}}(\mathcal{O})$.

Theorem (Seguin, 2024)

K *field of characteristic* 0

All K*-linear bijections* $f : K[X] \to K[X]$
preserving $\{a \in K[X] \text{ squarefree}\}$
are of the form $a \mapsto u \cdot \tau(a)$
with $u \in K^\times$ *and* $\tau \in \mathrm{Aut}_{K\text{-algebra}}(K[X])$.

Proof idea (for the number field result)

Global

f preserves
$\{a \in \mathcal{O} \text{ squarefree}\}$

$\longrightarrow$

Local

$f \bmod p^2$ preserves
$\{a \in \mathcal{O}/p^2\mathcal{O} \text{ squarefree}\}$
for all prime numbers p

$\downarrow$

$f \bmod p$ is of the
form $a \mapsto u \cdot \tau(a)$
for ∞ many p
with $u \in (\mathcal{O}/p\mathcal{O})^\times$,
$\tau \in \mathrm{Aut}_{\mathrm{ring}}(\mathcal{O}/p\mathcal{O})$

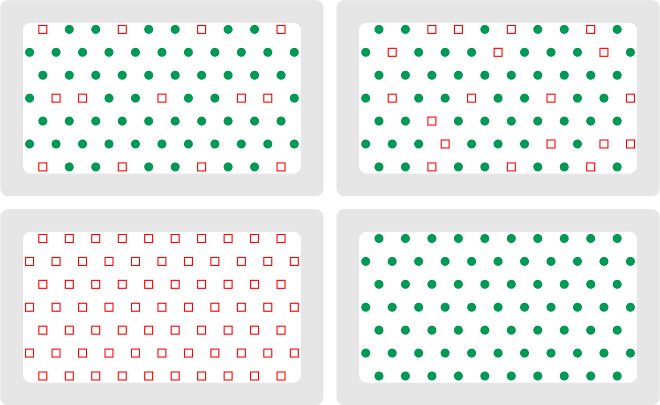
$\longleftarrow$

f is of the
form $a \mapsto u \cdot \tau(a)$
with $u \in \mathcal{O}^\times$,
$\tau \in \mathrm{Aut}_{\mathrm{ring}}(\mathcal{O})$

Shift spaces

$\mathbb{X}_{\mathcal{O}}$: set of maps $\mathcal{O} \to \{\bullet, \square\}$ that locally look like translates of:

$$a \mapsto \begin{cases} \bullet & a \text{ squarefree}, \\ \square & a \text{ not squarefree}. \end{cases}$$



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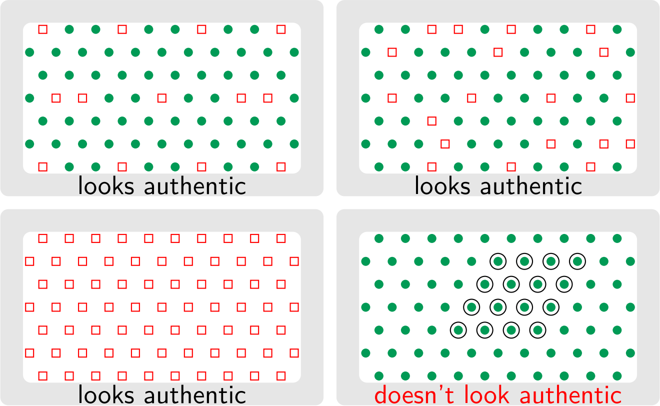


Dynamical systems

The additive group $\mathcal{O}$ acts on $\mathbb{X}_{\mathcal{O}}$ by translation, and we obtain a topological dynamical system $(\mathbb{X}_{\mathcal{O}}, \mathcal{O})$.

Theorem (Gundlach–Klüners, 2024; generalizing Baake–Bustos–Nickel, 2023)

(a) The extended symmetry group of $(\mathbb{X}_{\mathcal{O}}, \mathcal{O})$ is

$$\mathrm{ExSym}(\mathbb{X}_{\mathcal{O}}, \mathcal{O}) = \mathcal{O} \rtimes (\mathcal{O}^{\times} \rtimes \mathrm{Aut}_{\mathrm{ring}}(\mathcal{O})).$$

(b) If $(\mathbb{X}_{\mathcal{O}_1}, \mathcal{O}_1) \simeq (\mathbb{X}_{\mathcal{O}_2}, \mathcal{O}_2)$, then $\mathcal{O}_1 \simeq \mathcal{O}_2$.

Variants

One can replace $\{a \in \mathcal{O} \text{ squarefree}\}$ for example by:

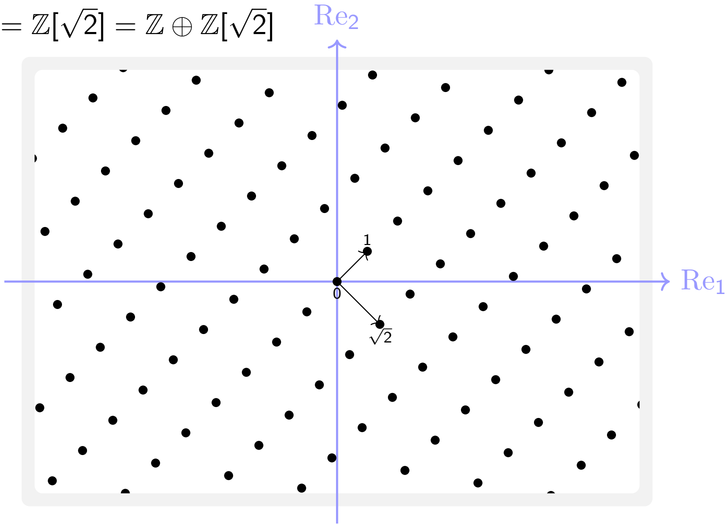
(a) $\{a \in \mathcal{O} : \forall \mathfrak{p} : a \not\equiv 0 \mod \mathfrak{p}^k\}$ for $k \geq 2$.

(b) $\{a \in \mathcal{O} : \forall \mathfrak{p} : a \not\equiv \ldots \mod \mathfrak{p}^{k(\mathfrak{p})}\}$ (often)

(c) $\{a \in \mathcal{O} \text{ prime}\}$

(d) $\mathcal{O}^\times$ if the number field is totally real

$$\mathcal{O} = \mathbb{Z}[\sqrt{2}] = \mathbb{Z} \oplus \mathbb{Z}[\sqrt{2}]$$



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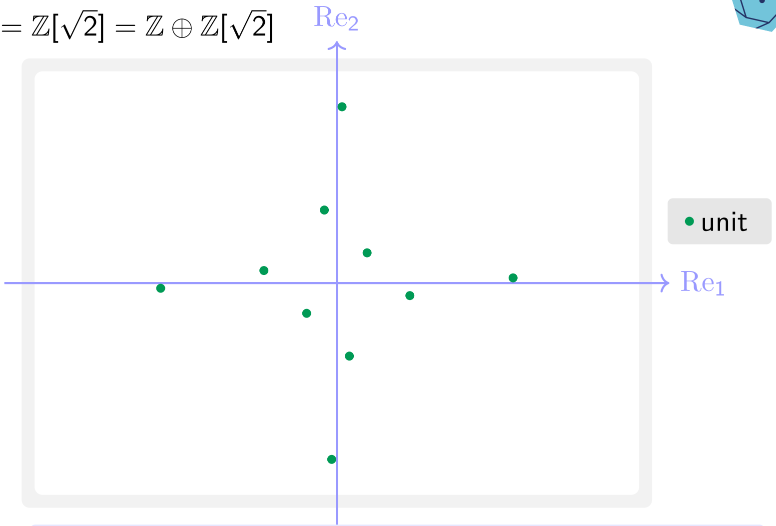

Affine $\mathbb{R}$-linear symmetries: $\underset{\mathcal{O}^\times}{(a \mapsto u \cdot a)} \underset{\rtimes}{\circ} \underset{\mathrm{Aut}_{\mathrm{ring}}(O)}{(a \mapsto \tau(a))}$

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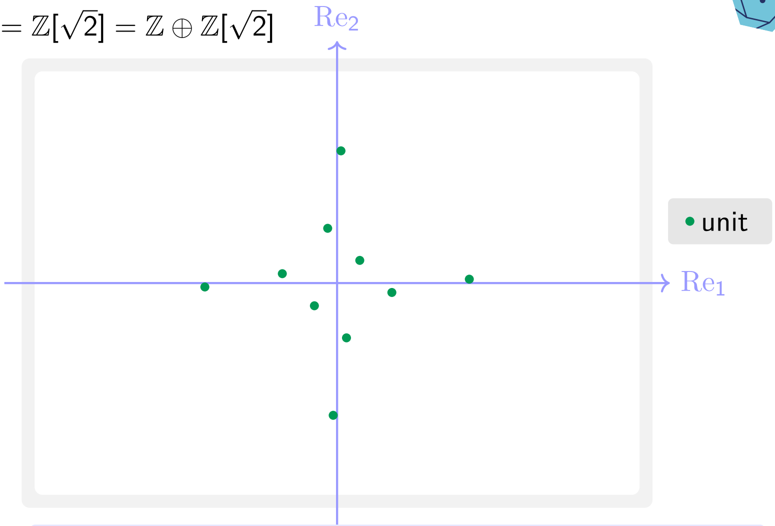

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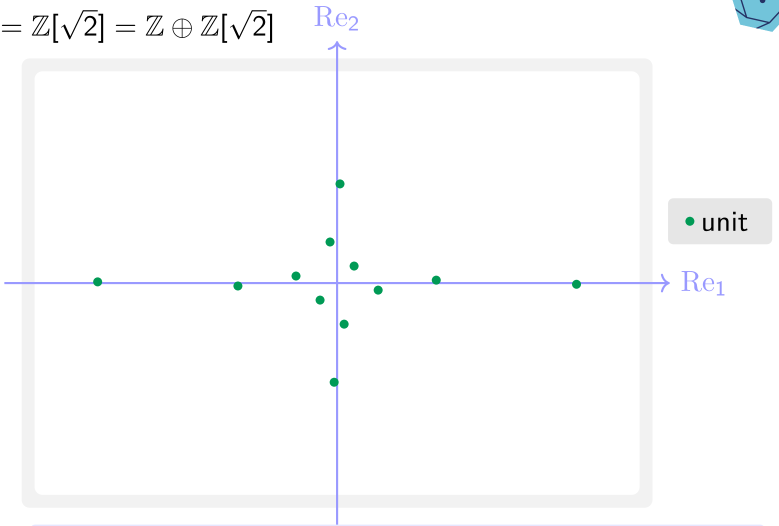

Affine $\mathbb{R}$-linear symmetries: $(a \mapsto u \cdot a) \circ (a \mapsto \tau(a))$

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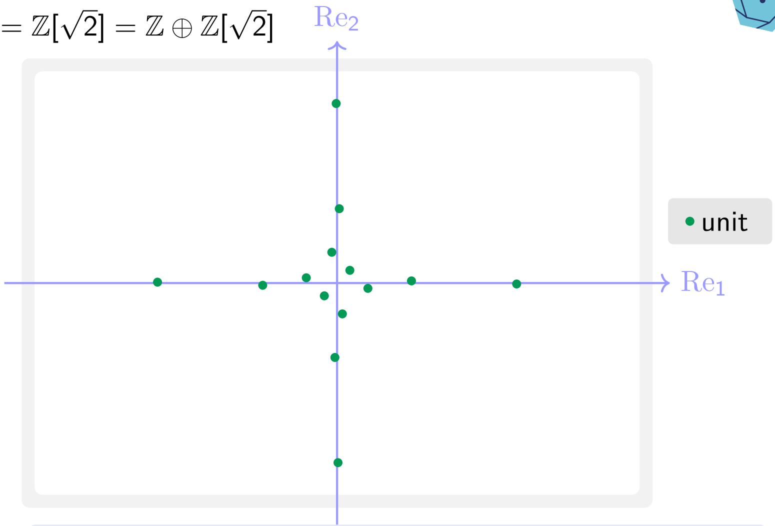

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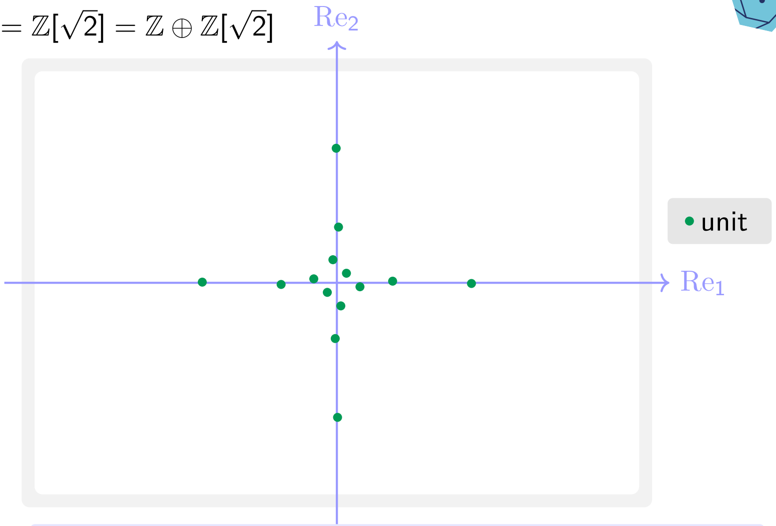

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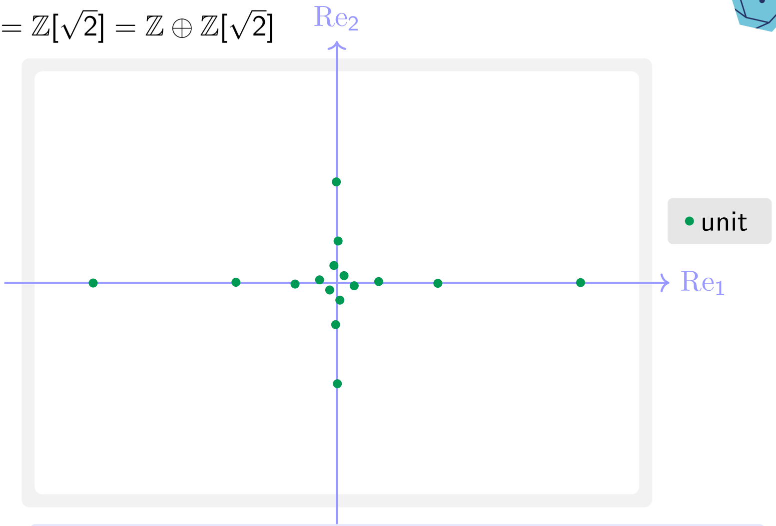

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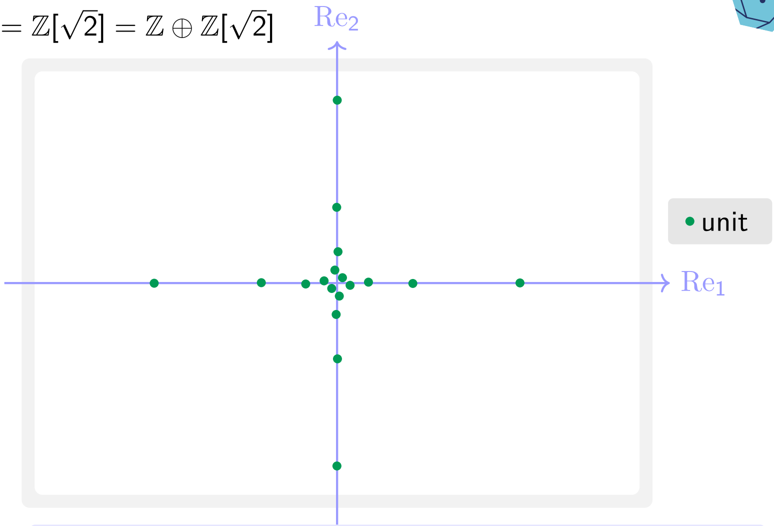

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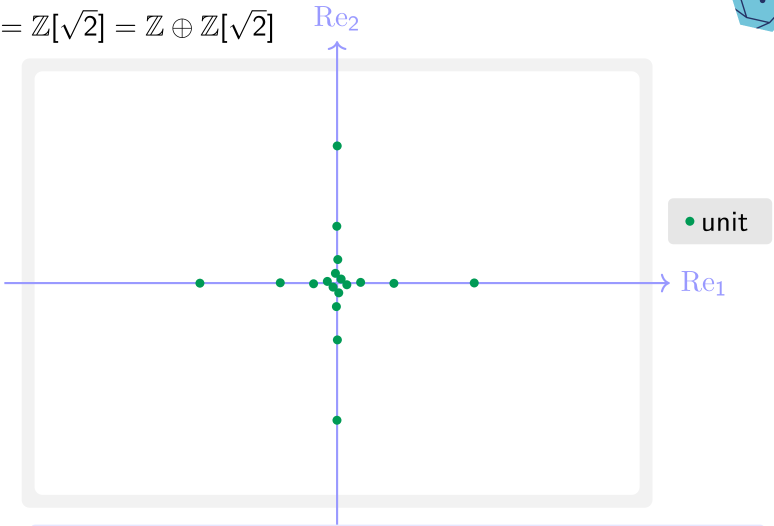

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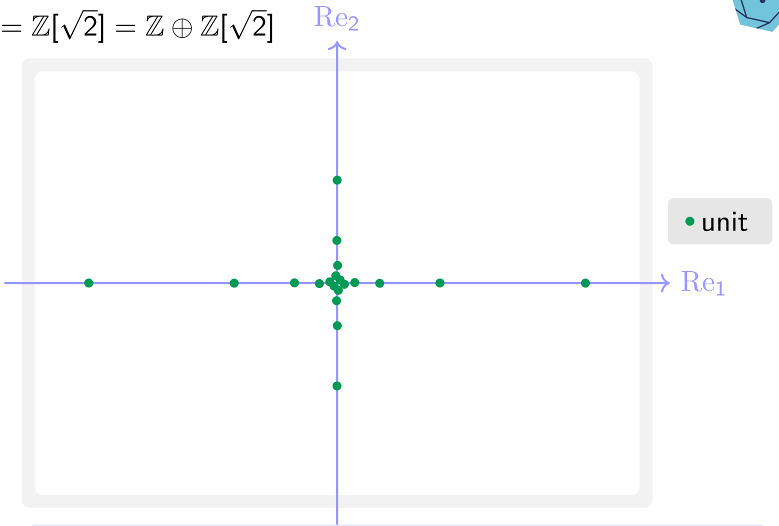

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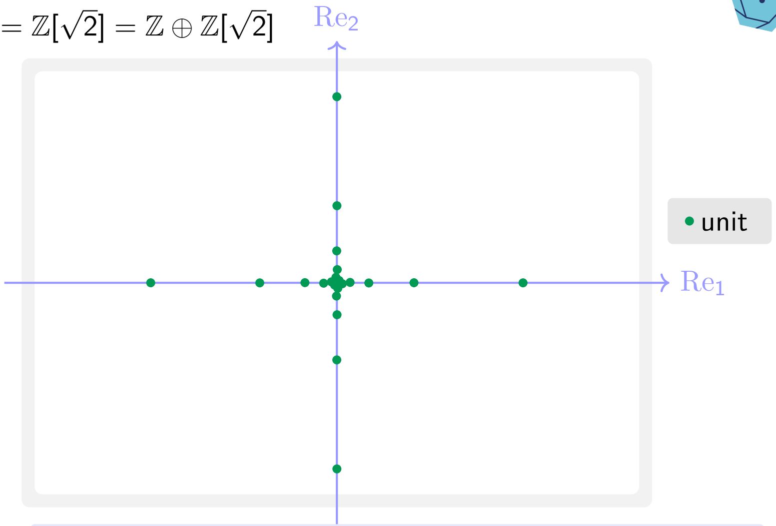

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