Sampling Cubic Rings

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2025-07-07

The LMFDB has a list of all . . .

- cubic number fields with $0 < \text{disc} < 3{,}375{,}000$
  (Belabas, Olivier)
- elliptic curves over $\mathbb{Q}$ with cond $\leq 500{,}000$
  (Cremona)
- zeros of $\zeta(s)$ with $0 < |\Im| \leq 30{,}610{,}046{,}000$
  (Platt)
- newforms with $\text{weight}^2 \cdot \text{level} \leq 4{,}000$
  (Best, Bober, Booker, Costa, Cremona, Derickx, Lowry-Duda, Lee, Roe, Sutherland, Voight)
- . . .

What if we want a (uniformly) random ...

- cubic number field with $0 < \text{disc} \leq 1{,}000{,}000{,}000{,}000{,}000{,}000$
- elliptic curve over $\mathbb{Q}$ with cond $\leq \ldots$
- zero of $\zeta(s)$ with $0 < |\Im| \leq \ldots$
- newform with weight$^2 \cdot$ level $\leq \ldots$
- ...

Warmup

How to pick an element of $\{1, 2, 3\}$ (uniformly) at random using coin flips?

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Method 1 (Rejection sampling)

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Average number of coin flips:
$O\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) = O(1)$.

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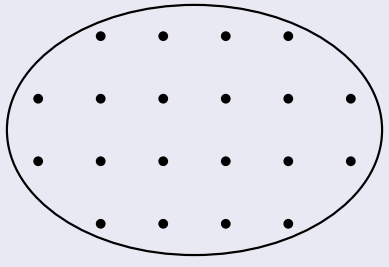

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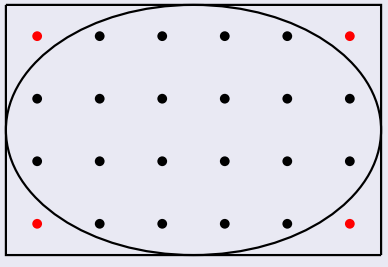

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Pick a real number $x \in [0, 1)$ uniformly at random.

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Expected running time: $O(1)$

Goal

Theorem (–)

There is an algorithm which computes a random cubic integral domain S with $0 < \mathrm{disc}(S) \leq T$ in expected time $\tilde{O}(\log T)$. Any such ring S is returned with probability proportional to $1/|\mathrm{Aut}(S)|$.

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Parameterization (Levi)

GL_2 acts on $V = \{f(X,Y) = aX^3 + bX^2Y + cXY^2 + dY^3\}$.

$$\begin{array}{ccc} \{\text{cubic ring } S\} & \longleftrightarrow & \mathrm{GL}_2(\mathbb{Z})\backslash\{f \in V(\mathbb{Z})\} \\ \mathrm{disc}(S) & = & \mathrm{disc}(f) \\ \mathrm{Aut}(S) & \simeq & \mathrm{Stab}(f) \\ \text{integral domain} & \Leftrightarrow & \text{irreducible (over } \mathbb{Z}) \end{array}$$

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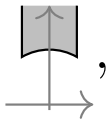
Want to produce . . .

a random orbit $\mathrm{GL}_2(\mathbb{Z})f$ in $V(\mathbb{Z})$ with $0 < \mathrm{disc}(f) \leq T$ and f irreducible, with probability proportional to $1/|\mathrm{Stab}(f)|$

Bhargava's trick

$$\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} 1/s & 0 \\ 0 & s \end{pmatrix} k \;\middle|\; (n, s^2) \in \text{  }, \; k \in O_2(\mathbb{R}) \right\}$$

fund. dom. for $\mathrm{GL}_2(\mathbb{Z})$ left acting on $\{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) = \pm 1\}$

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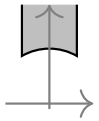
$$\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} 1/s & 0 \\ 0 & s \end{pmatrix} k \;\middle|\; (n, s^2) \in \text{  }, \; k \in O_2(\mathbb{R}) \right\}$$

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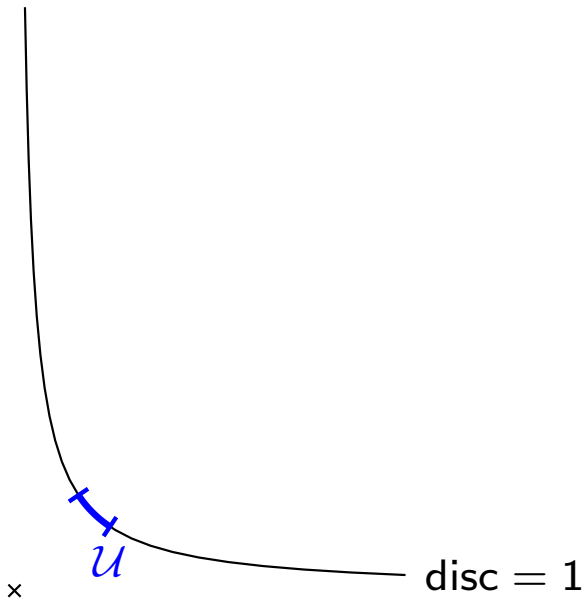

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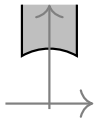
$$\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} 1/s & 0 \\ 0 & s \end{pmatrix} k \;\middle|\; (n, s^2) \in \text{  }, \; k \in O_2(\mathbb{R}) \right\}$$

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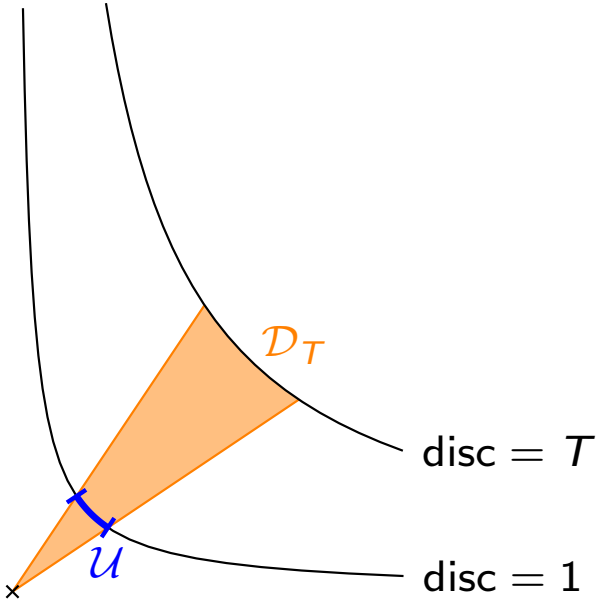

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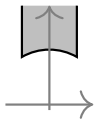
$$\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} 1/s & 0 \\ 0 & s \end{pmatrix} k \;\middle|\; (n, s^2) \in \text{  }, \; k \in O_2(\mathbb{R}) \right\}$$

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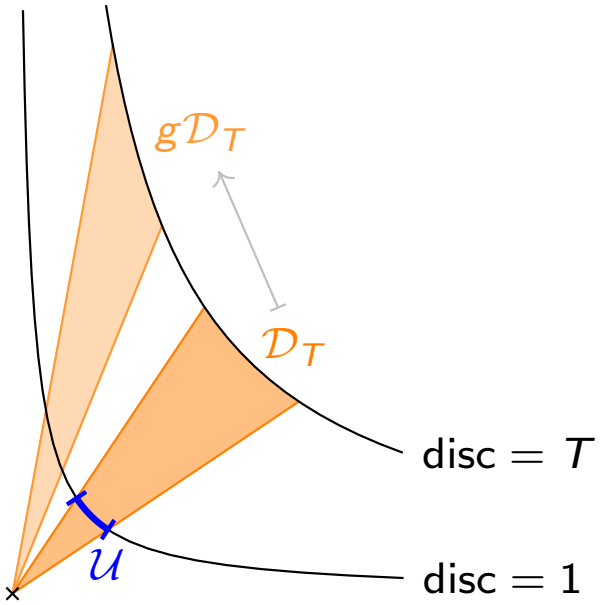

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There is a constant $C > 0$ such that for any orbit $\mathrm{GL}_2(\mathbb{Z})f$ in $V(\mathbb{Z})$,

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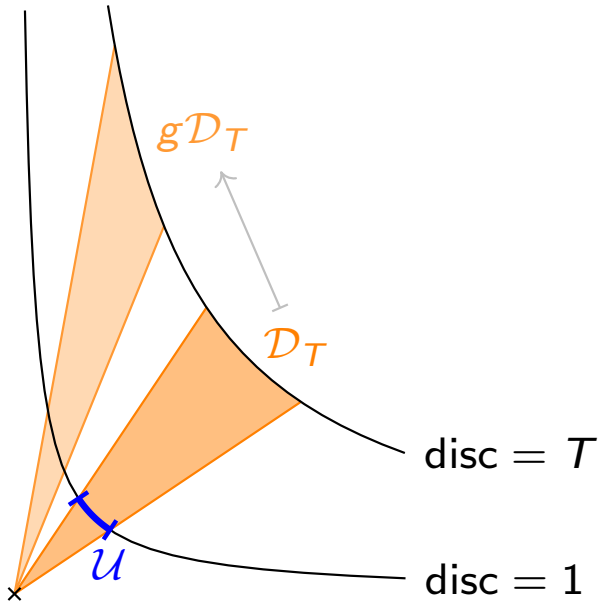
$$= \begin{cases} \frac{C}{|\mathrm{Stab}(f)|} & \text{if } 0 < \mathrm{disc}(f) \leq T, \\ 0 & \text{otherwise.} \end{cases}$$

Idea of the Algorithm



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To produce a random orbit $\mathrm{GL}_2(\mathbb{Z})f$ with $0 < \mathrm{disc}(f) \leq T$ and f irreducible:

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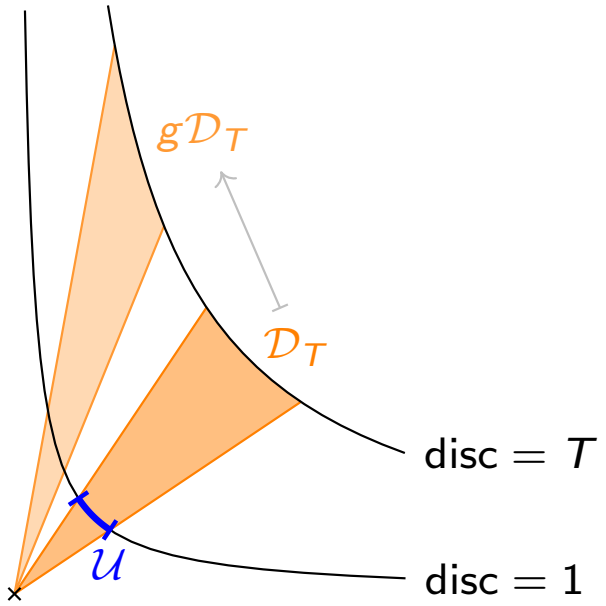

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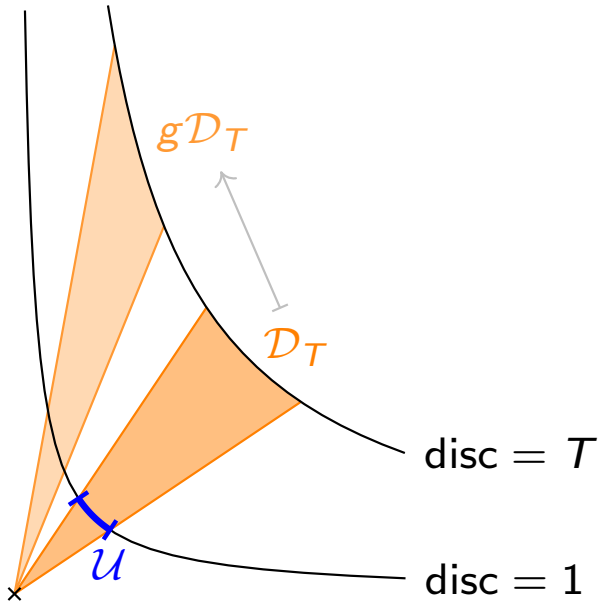

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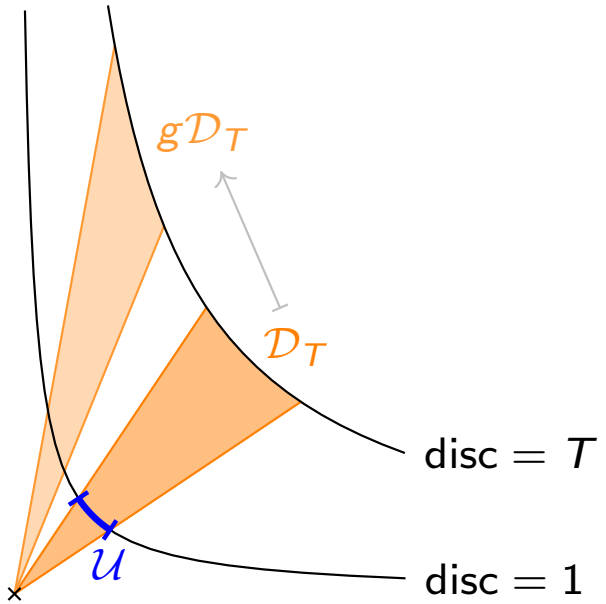

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$\Rightarrow$ Any particular orbit $\mathrm{GL}_2(\mathbb{Z})f$ will be returned with probability $\frac{C}{|\mathrm{Stab}(f)|} \cdot p_T$.

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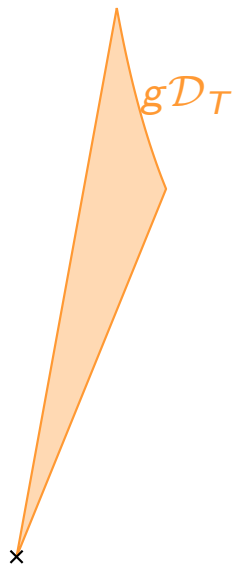

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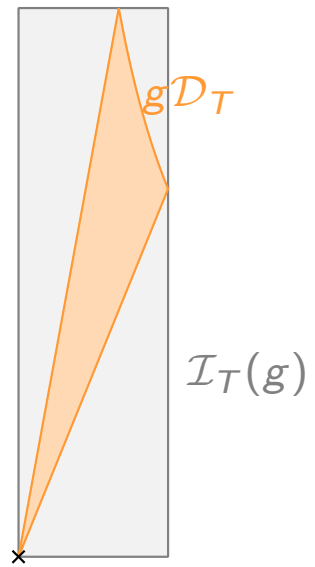
Let $\mathcal{I}_T(g)$ be the smallest axis-parallel box containing $g\mathcal{D}_T$.



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Let $\mathcal{I}_T(g)$ be the smallest axis-parallel box containing $g\mathcal{D}_T$.

If every $f \in \mathcal{I}_T(g) \cap V(\mathbb{Z})$ **has** X^3**-coefficient** $a = 0$**:**

$\Rightarrow$ There is no irreducible $f \in g\mathcal{D}_T \cap V(\mathbb{Z})$.

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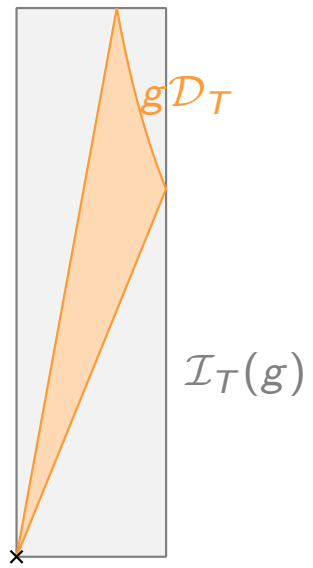

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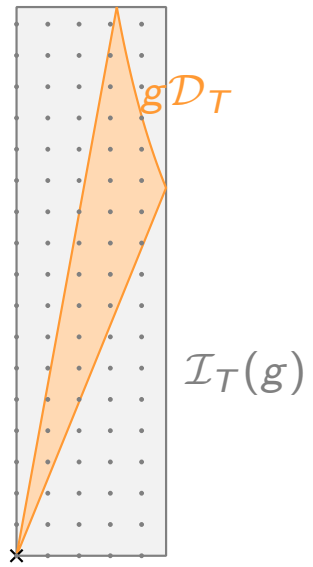

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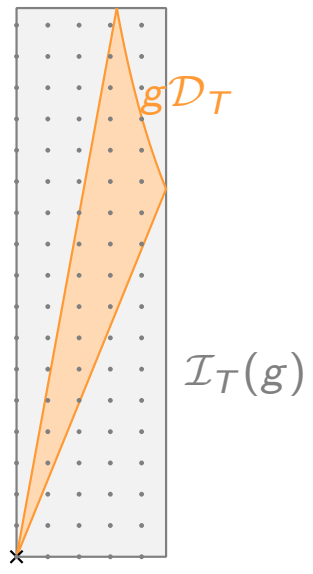

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Implementation of Step (2)

(1) Pick random g from $\mathcal{F}$ with probability measure $\mathrm{d}g$.

(2) Either return $\mathrm{GL}_2(\mathbb{Z})f$ for a random irreducible $f \in g\mathcal{D}_T \cap V(\mathbb{Z})$, or start over with (1). *Every* such element f should be picked with the *same* probability $p_T > 0$ *independent of* g.

Let $\mathcal{I}_T(g)$ be the smallest axis-parallel box containing $g\mathcal{D}_T$.

If every $f \in \mathcal{I}_T(g) \cap V(\mathbb{Z})$ **has** X^3**-coefficient** $a = 0$**:**

$\Rightarrow$ There is no irreducible $f \in g\mathcal{D}_T \cap V(\mathbb{Z})$.

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$\Rightarrow \#(\mathcal{I}_T(g) \cap V(\mathbb{Z})) \leq E_T$ for some constant E_T

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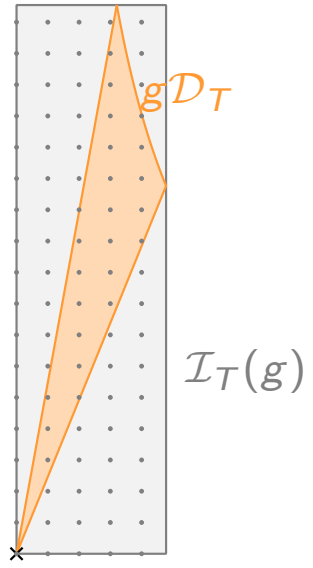

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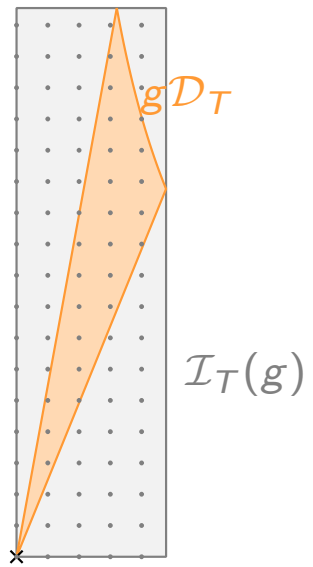

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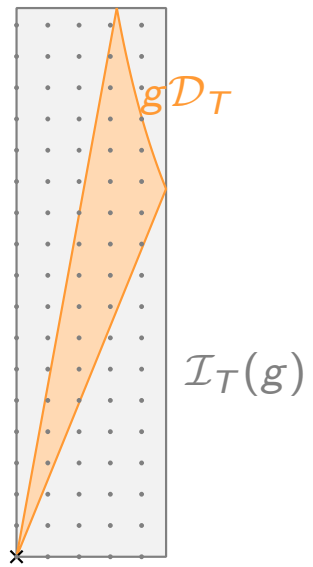

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5) Theoretically: quartic and quintic rings.