Representations of p-adic groups and applications

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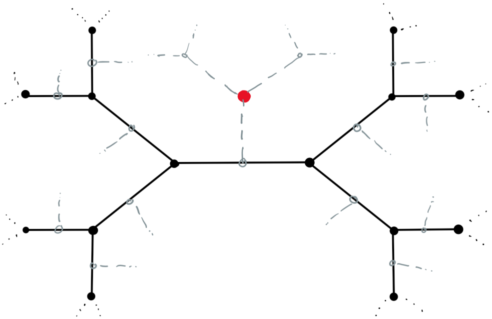
University of Cambridge and Duke University

September 2020



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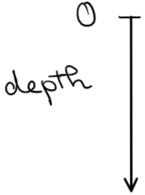

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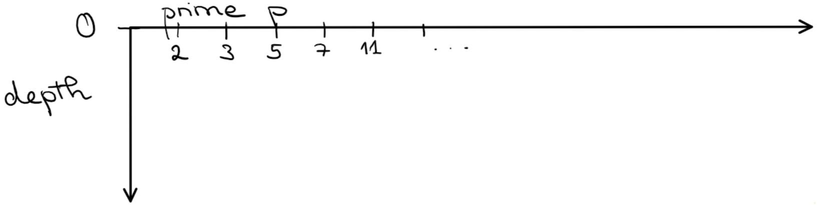

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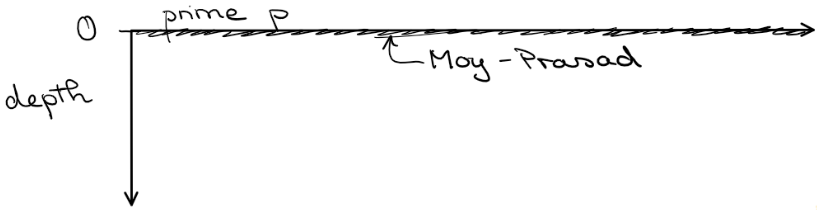

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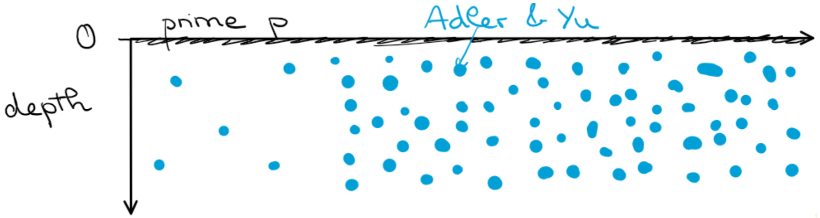

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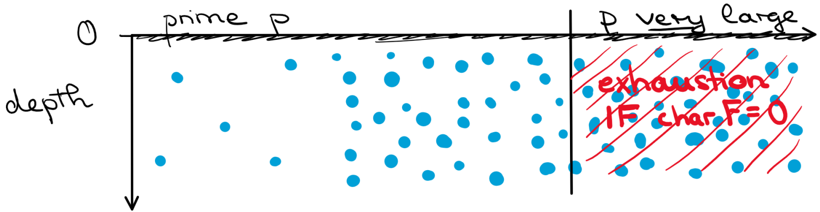

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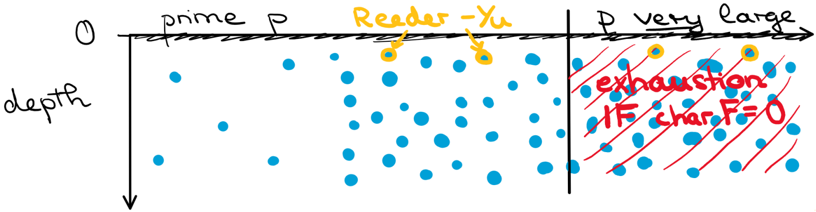

Epipelagic representations

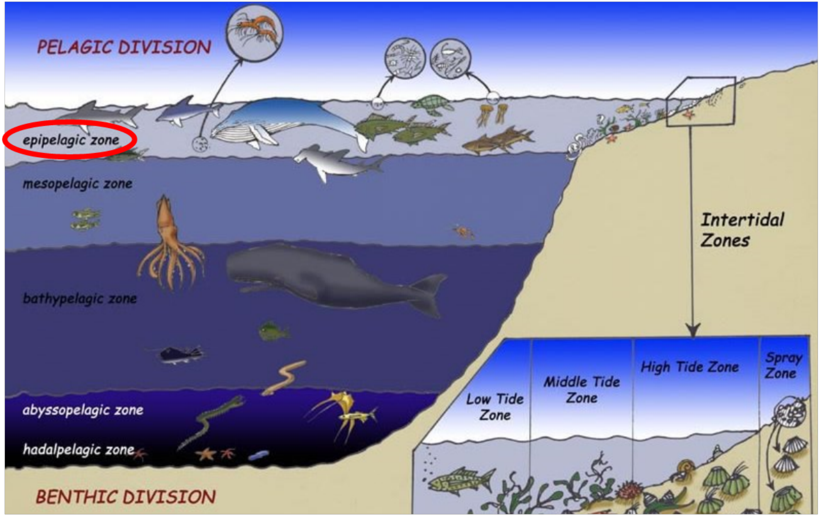

Figure: The epipelagic zone of the ocean;
source: Sheri Amsel. Glossary (what words mean) with pictures!. 2005-2015. April 2, 2015,
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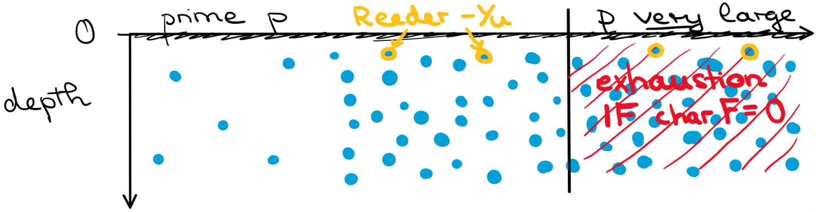

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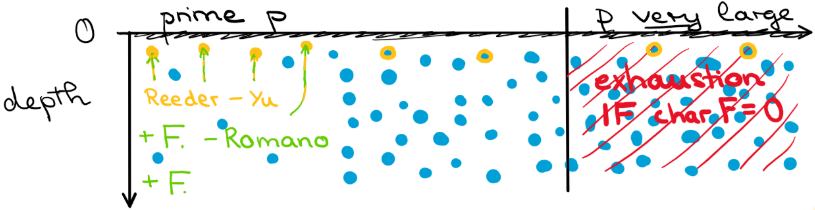

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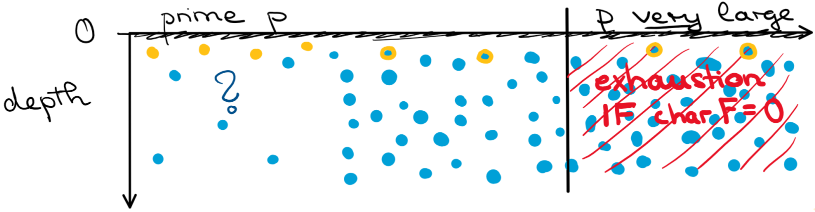

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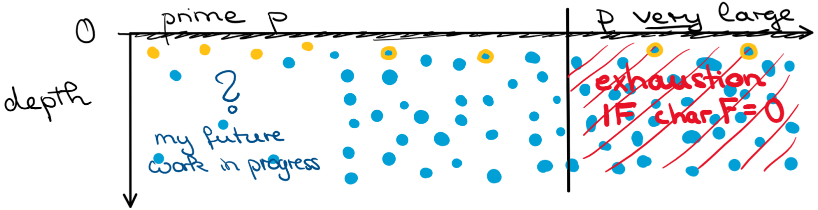

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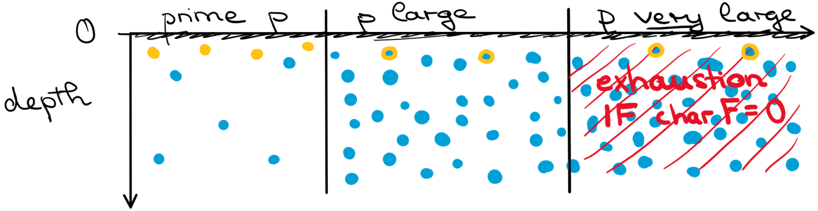

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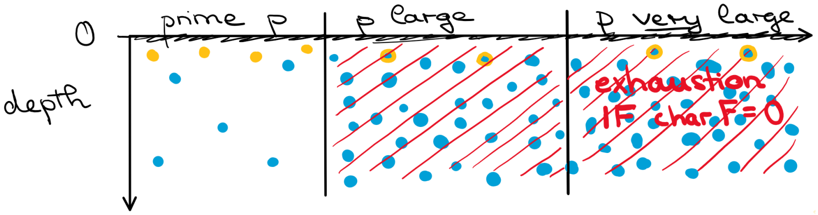

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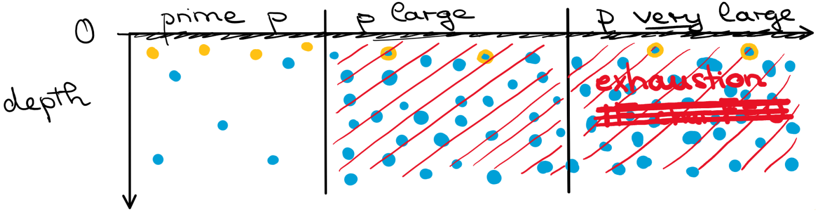

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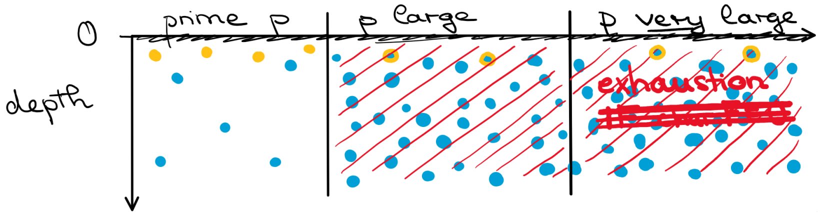
Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.



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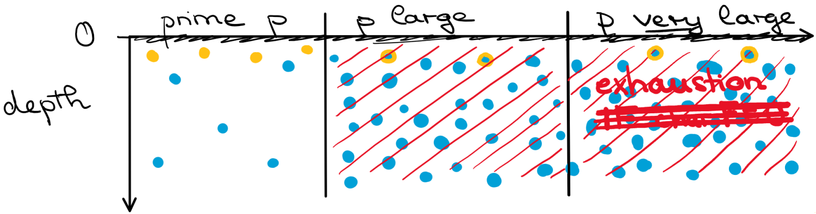
$\lvert W \rvert$	$(n+1)!$	$2^n \cdot n!$	$2^{n-1} \cdot n!$	$2^7 \cdot 3^4 \cdot 5$

type	E_7	E_8	F_4	G_2
$\lvert W \rvert$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2$	$2^2 \cdot 3$



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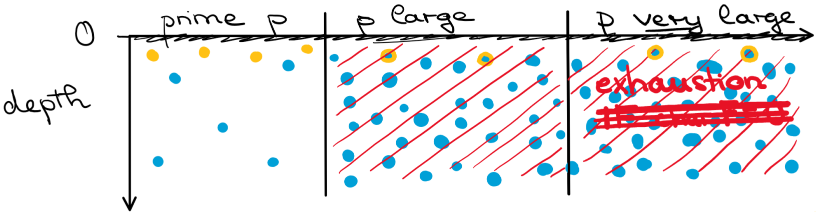

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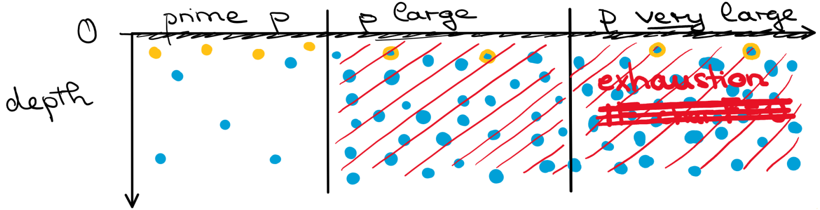

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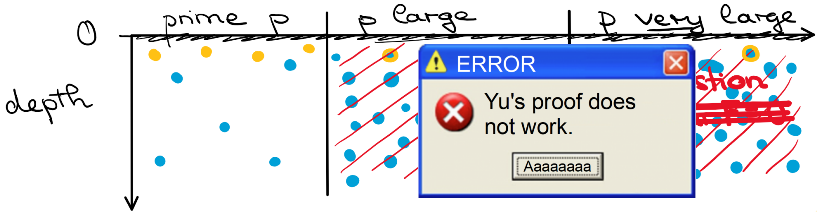

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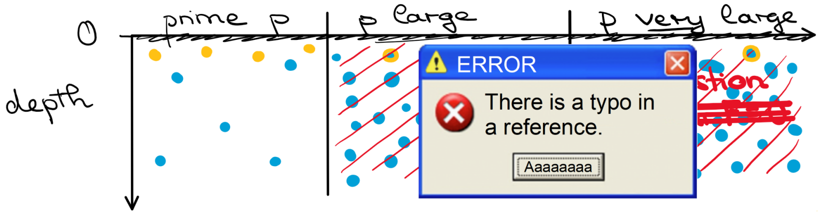

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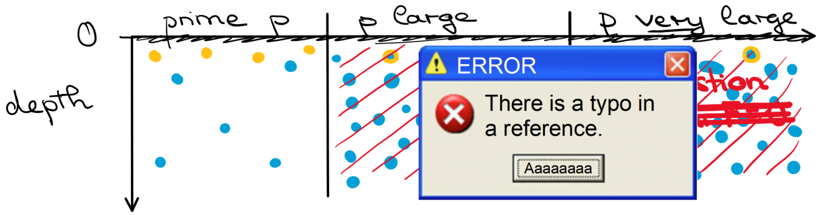

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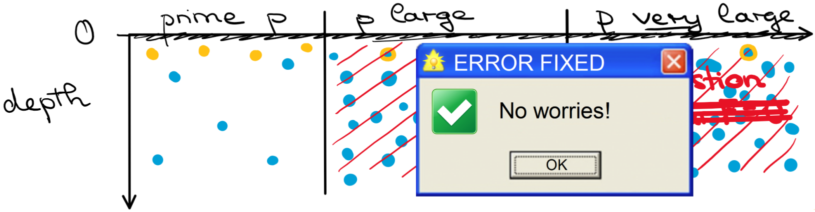

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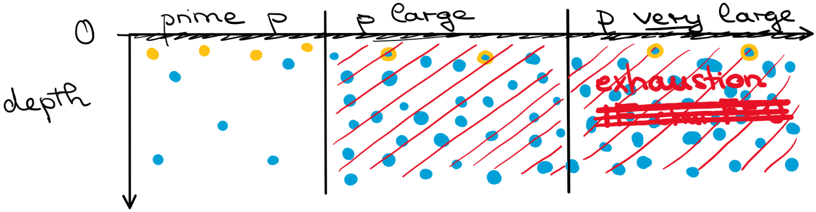

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Assume G splits over a tame extension, $\mathrm{char}(F) = 0$ and $p > \mathrm{Cox}(G)$. Then there exists a sequence $\{(U_m, \lambda_m)\}_{m \geqslant 1}$ such that

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a. $M(U_p U^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / U_p U^p \mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$
b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m - 1})$, suppose U_p acts smoothly on A_m smoothly:

Congruences of algebraic automorphic forms

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Let $p > \mathrm{Cox}(\mathcal{G})$. Then $\exists U_{p,m} \subset \mathcal{G}(\mathbb{Q}_p)$ with $U_{p,m} \circlearrowright A_m$

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Congruences of algebraic automorphic forms

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The end of the talk,
but only the beginning of the story ...