Dirichlet series

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Goal The goal of these notes is to briefly introduce the reader to the beautiful subject of Dirichlet series. In particular, we will answer some questions of the following form: given a (number theoretic) sequence $a_1, a_2, \ldots$ of nonnegative integers, what is the asymptotic behavior of $\sum_{n \leqslant N} a_n$ as N goes to infinity? More precisely, we want to find a "simple" function $F(N)$ so that $\sum_{n \leqslant N} a_n \sim F(N)$, by which we mean

$$\lim_{N \to \infty} \frac{\sum_{n \leqslant N} a_n}{F(N)} = 1.$$

Here are just a few of the many nice results that can be shown using Dirichlet series:

- $\sum_{n \leqslant N} d_n \sim N \log N$ if d_n denotes the number of divisors of n.
- $\sum_{n \leqslant N} \sigma_n \sim CN^2$ for some constant $C = 0.8224\ldots$ if σ_n denotes the sum of the divisors of n.
- $\sum_{n \leqslant N} \Delta_n \sim CN^\rho$ for some constants $C = 0.3181\ldots$ and $\rho = 1.728\ldots$ if Δ_n denotes the number of ordered factorizations of n into any number of factors bigger than 1.
- $\sum_{n \leqslant N} \mathbb{P}_n \sim \frac{N}{\log N}$ if $\mathbb{P}_n$ is 1 whenever n is a prime number and 0 otherwise. (This is the famous *prime number theorem.*)

A familiarity with basic facts of complex analysis (holomorphic functions, meromorphic functions, poles) is assumed in the later parts of these notes (the analytic statements).

Dirichlet series In combinatorics and additive number theory, it is often useful to associate to a sequence $a_0, a_1, \ldots$ of complex numbers the *ordinary generating function* $F(a, X) = \sum_{n=0}^{\infty} a_n X^n$, which is viewed as a formal power series. This means that we *define* the usual operations (addition, multiplication, derivative, . . .) on power series in the obvious way *even if the power series don't actually converge*:

$$F(a,X)+F(b,X)=\sum_{n=0}^{\infty}a_nX^n+\sum_{n=0}^{\infty}b_nX^n:=\sum_{n=0}^{\infty}(a_n+b_n)X^n=F(a+b,X)$$
$$F(a,X)\cdot F(b,X)=\left(\sum_{n=0}^{\infty}a_nX^n\right)\cdot\left(\sum_{n=0}^{\infty}b_nX^n\right):=\sum_{n=0}^{\infty}\Big(\sum_{\substack{x,y\geqslant 0\text{ s.t.}\\ x+y=n}}a_xb_y\Big)X^n=F(a*b,X)$$
$$\frac{\mathrm{d}}{\mathrm{d}X}F(a,X)=\frac{\mathrm{d}}{\mathrm{d}X}\sum_{n=0}^{\infty}a_nX^n:=\sum_{n=0}^{\infty}na_nX^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}X^n=F(a',X)$$

In multiplicative number theory, it is more useful to associate to a sequence $a_1, a_2, \ldots$ the *Dirichlet series* $D(a,s)=\sum_{n=1}^{\infty}a_nn^{-s}$. Again, we define the usual operations on Dirichlet series in the obvious way:

$$D(a,s)+D(b,s)=D(a+b,s)\qquad\text{where }(a+b)_n=a_n+b_n$$
$$D(a,s)\cdot D(b,s)=D(a*b,s)\qquad\text{where }(a*b)_n=\sum_{\substack{x,y\geqslant 1\text{ s.t.}\\ x\cdot y=n}}a_xb_y$$
$$\frac{\mathrm{d}}{\mathrm{d}s}D(a,s)=D(a',s)\qquad\text{where }a'_n=-a_n\log(n)$$

The addition and multiplication of Dirichlet series gives them the structure of a ring. The multiplicative identity is

$$1=D(\delta,s)\qquad\text{where }\delta=(1,0,0,\ldots).$$

A Dirichlet series $D(a,s)$ is invertible if and only if $a_1\neq 0$.

Let us look at another very simple sequence. Its Dirichlet series is called the *Riemann zeta function*:

$$\mathbb{1}=(1,1,\ldots)\qquad\longrightarrow\qquad D(\mathbb{1},s)=\sum_{n=1}^{\infty}n^{-s}=\zeta(s)$$

Despite the stupidity of this sequence, its Dirichlet series is very useful. Here are some examples of more interesting number-theoretic sequences. Note how each of the corresponding Dirichlet series can be written just in terms of the Riemann zeta function:

$$\left.\begin{aligned} d &= 1 * 1 \\ d_n &= \textstyle\sum_{xy=n} 1 \cdot 1 \\ &= \#\{(x,y) : xy = n\} \\ &= \#(\text{pos.) divisors of } n \end{aligned}\right\} \quad \longrightarrow \quad D(d,s) = D(\mathbb{1} * \mathbb{1}, s) = D(\mathbb{1}, s)^2 = \zeta(s)^2$$

$$\left.\begin{aligned} d^{(k)} &= \underbrace{\mathbb{1} * \cdots * \mathbb{1}}_{k \text{ times}} \\ d^{(k)}_n &= \textstyle\sum_{x_1 \cdots x_k = n} 1 \cdots 1 \\ &= \#\left\{\substack{(x_1, \ldots, x_k): \\ x_1 \cdots x_k = n}\right\} \end{aligned}\right\} \quad \longrightarrow \quad D(d^{(k)}, s) = D(\mathbb{1} * \cdots * \mathbb{1}, s) = D(\mathbb{1}, s)^k = \zeta(s)^k$$

$$\left.\begin{aligned} \text{id} &= (1, 2, 3, \ldots) \\ \text{id}_n &= n \end{aligned}\right\} \quad \longrightarrow \quad D(\text{id}, s) = \sum_{n=1}^{\infty} n \cdot n^{-s} = \sum_{n=1}^{\infty} n^{-(s-1)} = \zeta(s-1)$$

$$\left.\begin{aligned} \sigma &= \text{id} * \mathbb{1} \\ \sigma_n &= \textstyle\sum_{xy=n} x \cdot 1 \\ &= \textstyle\sum \text{divisors of } n \end{aligned}\right\} \quad \longrightarrow \quad D(\sigma, s) = D(\text{id}, s) D(\mathbb{1}, s) = \zeta(s-1)\zeta(s)$$

Complex analysis Let's now compare the complex analysis of power series and Dirichlet series.

Power series

The points X in the complex plane where a power series $F(a, X) = \sum_n a_n X^n$ converges are separated from the points where it doesn't by a circle $\{|\cdot| = r_c\}$ centered at the origin:[1]

- inside the circle, the power series converges (☺);
- outside the circle, the power series doesn't converge (☹);
- it might converge at some points on the circle (😕).

A power series is holomorphic (☺′) everywhere inside its radius of convergence.

Dirichlet series

The points s in the complex plane where a Dirichlet series $D(a, s) = \sum_n a_n n^{-s}$ converges are separated from the points where it doesn't by a vertical line $\{\Im = \sigma_c\}$:[2]

- to the right of the line, the Dirichlet series converges;
- to the left of the line, the Dirichlet series doesn't converge;
- it might converge at some points on the line.

A Dirichlet series is holomorphic everywhere to the right of its abscissa of convergence.

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[1] In general, the radius of convergence r_c could be 0, meaning that the power series converges only at $X = 0$, or could be ∞, meaning that the power series converges everywhere.

[2] In general, the abscissa of convergence σ_c could be ∞, meaning that the Dirichlet series doesn't converge anywhere, or could be $-\infty$, meaning that the Dirichlet series converges everywhere.

A power series converges absolutely (|☺|) inside its radius of convergence.

For Dirichlet series, the analogue statement doesn't hold. In general, there might be a vertical line $\{\Im = \sigma_{ac}\}$ separating the points of absolute convergence from the others.[3]

But if $a_n \geqslant 0$ for all n, the Dirichlet series converges absolutely to the right of σ_c.

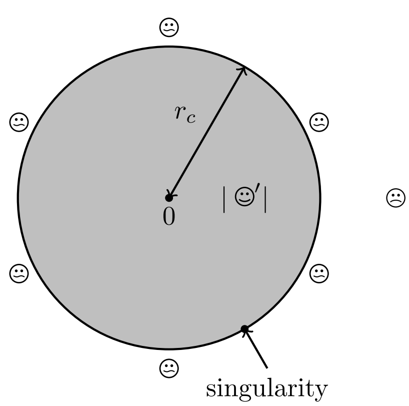
Another nice feature of power series is that they converge until they have a perfect excuse to stop: the function cannot be extended holomorphically to any larger circle centered at 0.

So if the power series has a meromorphic continuation, for example, there must be a pole on the circle of convergence.

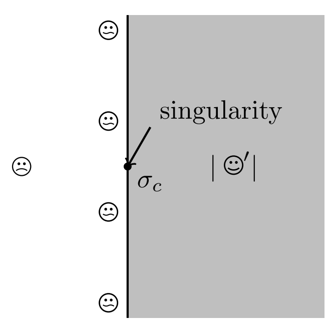
For Dirichlet series, this again fails in general: the function might have a holomorphic continuation to a wider strip $\{\Im \geqslant \sigma_c - \varepsilon\}$ for some $\varepsilon > 0$.

But if you assume that $a_n \geqslant 0$ for all n, the Dirichlet series must have a singularity at σ_c.

(a) Power series



(b) Dirichlet series if $a \geqslant 0$



Riemann zeta function What does the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ look like? It is easy to see that $\zeta(s)$ converges absolutely for $\Re(s) > 1$ and doesn't converge for $s = 1$ (which in fact means it never converges if $\Re(s) < 1$). Repeatedly integrating by parts, you can find a meromorphic continuation to the entire complex plane. There is only a simple pole at $s = 1$ with residue 1.

Since we can write $D(\mathbb{1}, s) = \zeta(s),\ D(d, s) = \zeta(s)^2,\ D(d^{(k)}, s) = \zeta(s)^k,\ D(\sigma, s) = \zeta(s-1)\zeta(s)$ all in terms of the Riemann zeta function, they can be meromorphically extended to the entire complex plane and we know their poles. But then

[3] $(\sigma_c \leqslant \sigma_{ac} \leqslant \sigma_c + 1)$

(since the sequences are nonnegative), we even know their respective abscissas of convergence: the series will converge until it "hits" the rightmost pole. This pole will be the abscissa of convergence.

Asymptotics for power series How can we make use of complex analysis of $F(a, X)$ to study the sequence a? Assume we know the radius of convergence r_c. In terms of the coefficients, $r_c = \liminf_{n\to\infty} \sqrt[n]{|a_n|}^{-1}$. This produces the upper bound $|a_n| \leqslant (r_c^{-1} + \varepsilon)^n$ for all $\varepsilon > 0$ and sufficiently large n. If we can find a nice sequence $b_0, b_1, \dots$ such that the radius of convergence r'_c of $F(a - b, X)$ is larger, we can get a (better) upper bound for the difference of the two sequences: $|a_n - b_n| \leqslant (r_c'^{-1} + \varepsilon)^n$.

For example, say $F(a, X)$ can be meromorphically continued to a circle of radius $r'_c > r_c$ and the continuation is holomorphic inside this circle except for a simple pole at $X = z_0$ with residue R (with $|z_0| = r_c$). Then, $F(a, X) - \frac{R}{X - z_0}$ is holomorphic inside the circle of radius r'_c. But we can write $\frac{R}{X-z_0} = -\frac{R/z_0}{1-X/z_0} = \sum_{n=0}^{\infty} -\frac{R}{z_0}\left(\frac{X}{z_0}\right)^n = F(b, X)$ with $b_n = -R/z_0^{n+1}$. We conclude that $a_n = -R/z_0^{n+1} + \mathcal{O}(r_c'^{-1} + \varepsilon)^n$ for all $\varepsilon > 0$. Note that (for small ε and large n) the error term is smaller than the main term.

To summarize, the main term of the asymptotics of a_n comes from singularities of $F(a, X)$ close to the origin. The further away the other singularities are, the smaller the error term.

Asymptotics for Dirichlet series Here is a theorem of a similar spirit concerning Dirichlet series:

Theorem 1 (Wiener–Ikehara)**.** *Assume $a_n \geqslant 0$ for all n and $\sigma_c > 0$ and that $\varphi(s) = (s - \sigma_c)^k D(a, s)$ has a holomorphic continuation to (a neighborhood of) $\{\Re \geqslant \sigma_c\}$ with $\varphi(\sigma_c) \neq 0$. Then*

$$\sum_{n \leqslant N} a_n \sim \frac{\varphi(\sigma_c)}{\sigma_c (k-1)!} N^{\sigma_c} \log^{k-1} N.$$

Let's apply this theorem to find asymptotics for the sequences we've seen before. We just need to find the rightmost pole σ_c of $D(a, s)$, its order k and its coefficient $\varphi(\sigma_c)$.

$$\begin{aligned} \mathbb{1} \quad &\longrightarrow \quad \zeta(s) \quad \longrightarrow \quad \text{simple pole at 1, coefficient 1} \\ &\longrightarrow \quad \sum_{n \leqslant N} 1 \sim N \end{aligned}$$

So the number of positive integers up to N is asymptotically roughly N. — Exciting, isn't it?

Let's see what happens for the other sequences:

$$\begin{aligned} d \quad &\longrightarrow \quad \zeta(s)^2 \quad \longrightarrow \quad \text{double pole at 1, coefficient 1} \\ &\longrightarrow \quad \sum_{n \leqslant N} d_n \sim N \log N \end{aligned}$$

$$\begin{aligned} d^{(k)} \quad &\longrightarrow \quad \zeta(s)^k \quad \longrightarrow \quad \text{order } k \text{ pole at 1, coefficient 1} \\ &\longrightarrow \quad \sum_{n \leqslant N} d_n^{(k)} \sim \frac{1}{(k-1)!} N \log^{k-1} N \end{aligned}$$

$$\begin{aligned} \sigma \quad &\longrightarrow \quad \zeta(s-1)\zeta(s) \quad \longrightarrow \quad \text{simple pole at 2, coefficient } \zeta(2) \\ &\longrightarrow \quad \sum_{n \leqslant N} \sigma_n \sim \frac{\zeta(2)}{2} N^2 \end{aligned}$$

These results can be easily shown directly without the use of Dirichlet series (and we can even get a nice error bound). [4]

Ordered factorizations Here is a simple sequence whose asymptotics can be analyzed substantially more easily using Dirichlet series.

Let $\Delta_n = \{(x_1, \dots, x_k) : k \geqslant 0,\ x_1, \dots, x_k \geqslant 2,\ x_1 \cdots x_k = n\}$ be the number of ordered factorizations of n into integers bigger than 1. Note that the number of factors is not fixed. [5]

We can write

$$\Delta_n = \sum_{k=0}^{\infty} \sum_{\substack{x_1, \dots, x_k \geqslant 2: \\ x_1 \cdots x_k = n}} 1 = \sum_{k=0}^{\infty} \sum_{\substack{x_1, \dots, x_k \geqslant 1: \\ x_1 \cdots x_k = n}} (1 - \delta_{x_1}) \cdots (1 - \delta_{x_k}).$$

This translates to

$$\Delta = \sum_{k=0}^{\infty} \underbrace{(\mathbb{1} - \delta) * \cdots * (\mathbb{1} - \delta)}_{k \text{ times}},$$

so

$$D(\Delta, s) = \sum_{k=0}^{\infty} D(\mathbb{1} - \delta, s)^k = \sum_{k=0}^{\infty} (\zeta(s) - 1)^k = \frac{1}{1 - (\zeta(s) - 1)} = \frac{1}{2 - \zeta(s)}.$$

[4] For example,

$$\begin{aligned} \sum_{n \leqslant N} \sigma_n &= \sum_{n \leqslant N} \sum_{ab = n} a = \sum_{ab \leqslant N} a = \sum_{b \leqslant N} \sum_{a \leqslant \lfloor N/b \rfloor} a = \sum_{b \leqslant N} \frac{1}{2} \left(\frac{N}{b} + \mathcal{O}(1) \right)^2 = \sum_{b \leqslant N} \left(\frac{N^2}{2b^2} + \mathcal{O}\left(\frac{N}{b} \right) \right) \\ &= \frac{\sum_{b=1}^{N} b^{-2}}{2} N^2 + \mathcal{O}(N \log N) = \frac{\zeta(2)}{2} N^2 + \mathcal{O}(N \log N). \end{aligned}$$

[5] This sequence's asymptotic behavior was studied by László Kalmár in the Hungarian article *A "factorisatio numerorum" problémájáról* (Matematikai és Fizikai Lapok) and in the German article *Über die mittlere Anzahl der Produktdarstellungen der Zahlen* (Acta Litt. Sci. Szeged).

(You can also verify this without using a summation over k by multiplying by $2-\zeta(s)$ and using a direct bijective argument.)

The poles of $D(\Delta, s)$ are the places where $\zeta(s) = 2$. It is clear that the function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is strictly decreasing on the real interval $(1, \infty)$. On this interval, the place ρ where the function is 2, is $\rho = 1.7286\ldots$. Furthermore, you can easily show that on each vertical line $\{\Re = \sigma\}$ with $\sigma > 1$, the function $\zeta(s)$ assumes the maximum value exactly at the intersection with the real axis, so ρ is in fact the (unique) rightmost (simple) pole of $D(\Delta, s)$. The Wiener–Ikehara theorem thus tells us that

$$\sum_{n \leqslant N} \Delta_n \sim -\frac{1}{\rho \zeta'(\rho)} N^{\rho}.$$

Multiplicative sequences, Euler products The sequences $\delta, \mathbb{1}, d, d^{(k)}, \sigma$ are all *multiplicative*: $a_{p_1^{e_1} \cdots p_k^{e_k}} = a_{p_1^{e_1}} \cdots a_{p_k^{e_k}}$ whenever $p_1, \ldots, p_k$ are distinct primes.[6]

For a multiplicative sequence a, we can rewrite $D(a, s)$ as an infinite product (called *Euler product*):

$$D(a, s) = \prod_{p \text{ prime}} \sum_{e=0}^{\infty} a_{p^e} p^{-es} = \prod_{p \text{ prime}} \left(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots\right).$$

(Try expanding this infinite product and using the uniqueness of factorizations!)

For example,

$$D(\mathbb{1}, s) = \prod_{p \text{ prime}} \sum_{e=0}^{\infty} p^{-es} = \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \cdots) = \prod_{p} \frac{1}{1 - p^{-s}}$$

$$D(d, s) = \prod_{p \text{ prime}} \sum_{e=0}^{\infty} (e+1) p^{-es} = \prod_{p \text{ prime}} (1 + 2p^{-s} + 3p^{-2s} + \cdots) = \prod_{p} \frac{1}{(1 - p^{-s})^2}$$

$$D(d^{(k)}, s) = \prod_{p} \frac{1}{(1 - p^{-s})^k}$$

$$D(\sigma, s) = \prod_{p \text{ prime}} \sum_{e=0}^{\infty} (1 + p + \cdots + p^e) p^{-es}$$

$$= \prod_{p \text{ prime}} (1 + (1+p)p^{-s} + (1 + p + p^2)p^{-2s} + \cdots) = \prod_{p} \frac{1}{(1 - p^{1-s})(1 - p^{-s})}$$

The prime number theorem Let's find an asymptotic formula for the number of primes up to N. Unfortunately, the Dirichlet series for the sequence

$$\mathbb{P}_n = \begin{cases} 1, & n \text{ prime} \\ 0, & \text{else} \end{cases}$$

[6] Equivalently, $a_1 = 1$ and $a_{nm} = a_n a_m$ whenever n and m are coprime. Note that not necessarily $a_{p^2} = a_p^2$.

doesn't have a simple expression in terms of the Riemann zeta function.

But the Euler product expansions suggest a different approach. We want a sum over primes, not a product, so it might be helpful to look at the *logarithm* of $\zeta(s)$:

$$\log \zeta(s) = \sum_{p \text{ prime}} \log \frac{1}{1-p^{-s}} = \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}$$

In complex analysis, logarithms can be somewhat nasty: It's impossible to define $\log(s)$ in a neighborhood of the origin, unless you look at a branched cover of the complex plane. Therefore, if a function $f(s)$ has zeros, $\log f(s)$ is defined only on a branched cover of the complex plane.

To circumvent this issue, we look at the derivative $\frac{\mathrm{d}}{\mathrm{d}s} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)}$. This quotient is of course meromorphic everywhere in the complex plane.

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{\mathrm{d}}{\mathrm{d}s} \log \zeta(s) = \sum_{p \text{ prime}} \sum_{k=1}^{\infty} (-\log p) p^{-ks}$$

Thus, we have found an expression for the Dirichlet series of the sequence

$$\Lambda_n = \begin{cases} \log p, & n = p^k \text{ for some prime } p \text{ and some } k \geqslant 1, \\ 0, & \text{else.} \end{cases}$$

$$D(\Lambda, s) = -\frac{\zeta'(s)}{\zeta(s)}$$

The poles of this function are at the places where $\zeta(s)$ has a pole ($s = 1$) or a zero. Already knowing that $\zeta(s)$ has no zero with $\Re \geqslant 1$ is enough to apply the Wiener–Ikehara theorem and conclude that

$$\sum_{n \leqslant N} \Lambda_n \sim N.$$

The left-hand side is clearly closely related to the number of primes up to N and it is in fact not too hard to deduce the famous prime number theorem:[7]

$$\#(\text{primes} \leqslant N) \sim \int_2^N \frac{\mathrm{d}t}{\log t} \sim \frac{N}{\log N}.$$

By analogy with asymptotics in the case of power series, you should expect that the further away from the line $\{\Re = 1\}$ the zeros of $\zeta(s)$ (the poles of $D(\Lambda, s)$ other than $s = 1$) are, the smaller the error in this asymptotic should be. In fact, the Riemann hypothesis states that the only zeroes of $\zeta(s)$ are the negative even integers and some (infinitely many) numbers on the line $\{\Re = \frac{1}{2}\}$.

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[7] Hint:

$$N \sim \sum_{n \leqslant N} \Lambda_n \approx \sum_{n \leqslant N(1-\varepsilon)} \Lambda_n + \log N \sum_{N(1-\varepsilon) < n \leqslant N} \mathbb{P}_n \sim N(1-\varepsilon) + \log N \sum_{N(1-\varepsilon) < n \leqslant N} \mathbb{P}_n.$$

Further reading For a deeper introduction to Dirichlet series and far more examples, I highly recommend the book *Problems in Analytic Number Theory* by M. Ram Murty.