Math 280Y: Arithmetic Statistics

Spring 2023

Problem set #7

due Sunday, April 30 at 10pm

Problem 1. Consider the trivial cubic extension $S = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ of $\mathbb{Z}$. Find all cubic subextensions $S' \subset S$ of $\mathbb{Z}$ of index $[S : S'] \in \{p, p^2, p^3\}$, where p is prime.
Hint: Use the appropriate normal form

Problem 2. For any cubic form $f \in \mathcal{V}(\mathbb{Z}_p)$, let $\omega_p(f)$ be the number of roots of f in $\mathbb{P}^1(\mathbb{F}_p)$. Show:

$$\sum_{\substack{[f]\in \mathrm{GL}_2(\mathbb{Z})\backslash\mathcal{V}(\mathbb{Z}_p)\\ \mathrm{disc}(f)=T\\ f\notin\mathcal{V}^m(\mathbb{Z}_p)}} 1 = \sum_{\substack{[f]\in \mathrm{GL}_2(\mathbb{Z})\backslash\mathcal{V}(\mathbb{Z}_p)\\ \mathrm{disc}(f)=T/p^2}} \omega_p(f) - \sum_{\substack{[f]\in \mathrm{GL}_2(\mathbb{Z})\backslash\mathcal{V}(\mathbb{Z}_p)\\ \mathrm{disc}(f)=T/p^4}} (\omega_p(f)-1)$$

Problem 3. Let $n \geqslant 2$ be prime. Show that there is a constant $C > 0$ such that

$$\#\{L \text{ Galois extension of } \mathbb{Q} \text{ with Galois group } C_n \mid |\mathrm{disc}(L)| \leqslant T\} \sim C{\cdot}T^{1/(n-1)}$$

for $T \to \infty$.