Math 280Y: Arithmetic Statistics

Spring 2023

Problem set #3

due Friday, February 24 at 10pm

Problem 1 (Cauchy–Davenport theorem)**.** Let p be a prime number and let $\varnothing \neq A, B \subseteq \mathbb{Z}/p\mathbb{Z}$. Show that the set $A + B$ of sums $a + b$ with $a \in A$ and $b \in B$ has size

$$\#(A + B) \geqslant \min(\#A + \#B - 1, p).$$

Problem 2. Let f_1 and f_2 be fundamental domains for the action of G on X.

a) Show that $\sum_{x \in X} f_1(x) = \sum_{x \in X} f_2(x)$.

b) Assume that G is countable, that X is a measure space, and that the action is measure-preserving. Show that $\int_X f_1(x)\mathrm{d}x = \int_X f_2(x)\mathrm{d}x$.

Hint: Consider the functions $l_g(x) = f_1(x)f_2(gx)$ for $g \in X$.

Problem 3. Assume that G is countable, that X is a measure space, and that the action of G on X is measure-preserving. Let S be a measurable elephantamental domain for this action.

a) Show that $\int_X f(x)\mathrm{d}x \leqslant \mathrm{vol}(S)$.

b) Show that $\int_X f(x)\mathrm{d}x > 0$ if $\mathrm{vol}(X) > 0$.

c) Show that there is a countable group G with a measure-preserving action on a measure space X whose stabilizers are all finite but which doesn't have a measurable fundamental domain.

Problem 4. Order the full lattices $\Lambda \subseteq \mathbb{Z}^n \subseteq \mathbb{R}^n$ by their covolume.

a) Show that as $T \to \infty$,

$$\#\{\Lambda \subseteq \mathbb{Z}^n \text{ full lattice with } \operatorname{covol}(\Lambda) \leqslant T\} \sim \frac{1}{n}\zeta(2)\cdots\zeta(n) \cdot T^n.$$

b) Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{Z}^n$. Show that

$$\mathbb{P}_{\Lambda \subseteq \mathbb{Z}^n \text{ full lattice}}(e_1 \in \Lambda) = 0.$$

c) Let $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ be the projection onto the first $n-1$ coordinates. Then, $\pi(\Lambda) \subseteq \mathbb{Z}^{n-1}$ is always a full lattice. Show that

$$\mathbb{P}_{\Lambda \subseteq \mathbb{Z}^n \text{ full lattice}}(\pi(\Lambda) = \mathbb{Z}^{n-1}) = \frac{1}{\zeta(2) \cdots \zeta(n)}.$$

Problem 5 (Mahler's criterion, bonus)**.** Equip $\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R})$ with the quotient topology. Let X be a closed subset of $\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R})$. Show that X is compact if and only if there exist $0 < C \leqslant C' < \infty$ such that the successive minima $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ of any lattice Λ corresponding to an element of X satisfy $C \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant C'$.
Hint: Use the Iwasawa decomposition and Siegel's almost fundamental domain.