Math 280Y: Arithmetic Statistics

Spring 2023

Problem set #1

due Friday, February 3 at 10pm

Problem 1. a) Describe an ordering $\mathrm{inv} : \mathbb{N} \to \mathbb{R}$ for which

$$\mathbb{P}_{x\in\mathbb{N}}(x \text{ even}) = 0.$$

b) Order pairs $(x, y) \in \mathbb{N}^2$ by $\max(x, y)$. What is

$$\mathbb{P}_{(x,y)\in\mathbb{N}^2}(\gcd(x, y) = 1)?$$

Problem 2. a) We say that a partition of a set X has *type* $(k_1, \ldots, k_r)$ if it consists of r subsets of X of sizes $k_1, \ldots, k_r$. Compute the number of partitions of $\{1, \ldots, n\}$ of type $(k_1, \ldots, k_r)$.

b) Let $X = \{1, \ldots, n\}$ and consider a group operation $\cdot : X \times X \to X$, chosen uniformly at random. Show that there is a constant C_n such that for all groups G of size n, we have

$$\mathbb{P}_{\cdot \text{ group operation on } X}((X, \cdot) \cong G) = C_n \cdot \frac{1}{\#\operatorname{Aut}(G)}.$$

Problem 3. Let $N(T)$ be the number of quadratic number fields K (up to isomorphism) with $|\operatorname{disc}(K)| \leqslant T$.

a) Show that for $T \to \infty$, we have

$$N(T) \sim \prod_p \left(1 - \frac{1}{p^2}\right) \cdot T$$

b) Show that if we order the quadratic number fields by $|\operatorname{disc}(K)|$, then for any odd prime ℓ,

$$\mathbb{P}_{K \text{ quadratic number field}}(K \text{ ramified at } \ell) = \frac{1}{\ell + 1}.$$

Problem 4. a) Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial. Show that

$$\mathbb{E}_{p \text{ prime}}(\#\{x \in \mathbb{F}_p \mid f(x) = 0\}) = 1.$$

b) Let $n \geqslant 2$. Show that the number of squarefree monic polynomials $f(X) \in \mathbb{F}_q[X]$ of degree n is $q^n - q^{n-1}$. (Hint: Every monic polynomial $a(X)$ can be written uniquely as $a(X) = f(X)g(X)^2$, where $f(X)$ is squarefree and both $f(X)$ and $g(X)$ are monic.)