# Math 229: Introduction to Analytic Number Theory

## Spring 2022

### Problem set #6

due Friday, March 25 at noon

**Problem 1.** Show that

$$\zeta(\sigma + it) \ll (1 + |t|^{1-\sigma}) \cdot \min(\frac{1}{|1-\sigma|}, \log |t|)$$

if $2 \geq \sigma \geq \frac{1}{2}$ and $|t| \geq 2$.
**Hint:** Use the Euler–Maclaurin formula on an interval $[N, \infty)$ and choose $N$ wisely depending on $\sigma$ and $t$.

**Problem 2.** As in class, write

$$-\frac{\Gamma'}{\Gamma}(s) = B_\Gamma + \frac{1}{s} + \sum_{n=1}^{\infty} \left( \frac{1}{s+n} - \frac{1}{n} \right)$$

and

$$\frac{1}{s} + \frac{1}{s-1} + \frac{\xi'}{\xi}(s) = B + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

and for any primitive nontrivial character $\chi$ modulo $q$, write

$$\frac{\xi'}{\xi}(s, \chi) = B_\chi + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Let $\gamma$ be the Euler–Mascheroni constant.

a) Show that $B_\Gamma = \gamma$ and $B = \frac{1}{2}\log(4\pi) - 1 - \frac{1}{2}\gamma$.

b) Show that

$$B = -\sum_{\rho} \Re(1/\rho)$$

and

$$\Re(B_\chi) = -\sum_{\substack{\rho \text{ zero} \\ \text{of } \xi(s,\chi)}} \Re(1/\rho).$$

**Problem 3.** a) Show that there is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that for all $s \in \mathbb{C}$, we have

$$s(s-1)\xi(s) = f((s-1/2)^2).$$

b) What is the order of $f(s)$?

c) Show that

$$s(s-1)\xi(s) = A \cdot \prod_{\substack{\rho \text{ nontrivial zero of } \zeta(s) \\ \text{with } \Im(\rho) > 0}} \left(1 - \left(\frac{s-1/2}{\rho-1/2}\right)^2\right)$$

for some constant $A$. (You may assume that $\zeta(s)$ has no real nontrivial zero.)