Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #5

due Thursday, March 10 at noon

Problem 1. a) Show that $\zeta(1-n) = (-1)^{n+1} B_n(0)/n$ for $n \geq 1$.
Hint: Use the Euler–Maclaurin formula on an interval $[\varepsilon, N]$ and let $\varepsilon \to 0$ and $N \to \infty$.

b) Show that

$$\zeta(n) = \tfrac{1}{2}(2\pi)^n (-1)^{n/2+1} B_n(0)/n!$$

for even integers $n \geq 2$.
Hint: Use the functional equation and properties of the gamma function.

c) (bonus) Let χ be the nontrivial character modulo 4. Show that

$$L(1, \chi) = \pi/4.$$

Hint: There are several ways of doing this: Either follow the strategy from parts a) and b), or use the class number formula.

Problem 2. Consider the multiplicative characters χ modulo a fixed integer $q \geq 1$. Show that

$$\prod_{\chi} L(s, \chi) = \prod_{p \nmid q \text{ prime}} \left(1 - p^{-\operatorname{ord}_q(p)s}\right)^{-\varphi(q)/\operatorname{ord}_q(p)},$$

where we denote by $\operatorname{ord}_q(n)$ the multiplicative order of the residue class $n \bmod q$.

Problem 3. Let $q \geq 1$ and let $c : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Q}_{\geq 0}$ be any function. Assume that $b := \frac{1}{\varphi(q)} \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^\times} c(x) > 0$. For an integer $n \geq 1$, let $a_n = \prod_{p|n} c(p \bmod q)$. Show that

$$\sum_{n \leq X} a_n \sim C \cdot X (\log X)^{b-1},$$

for some constant $C > 0$.

Problem 4. Let χ be a nontrivial real character modulo q. Assume that q is prime. Let V be the $\mathbb{C}$-vector space of functions $\mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$. Let $T : V \to V$ be the map sending a function c to its Fourier transform $\widehat{c}$ given by $\widehat{c}(y) = \sum_{x \in \mathbb{Z}/q\mathbb{Z}} c(x) e^{2\pi i x y/q}$.

a) Show that $\chi(x) = (\frac{x}{q})$ for all x.

b) Show that $\tau(\chi) := \widehat{\chi}(1)$ is the trace of the linear map T.

c) What are the eigenvalues of T^2 and what are their multiplicities?

d) Show that $\det(T) = i^{q(q-1)/2} \cdot q^{q/2}$.
Hint: Use trigonometry to determine the sign of $\det(T)$.

e) Show that $\tau(\chi) = \sqrt{q}$ if χ is even and $\tau(\chi) = i\sqrt{q}$ if χ is odd.
Hint: We already know this except for the sign.