Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #4

due Wednesday, February 23 at noon

Although we haven't finished the proof, you may use the Wiener–Ikehara theorem and Kato's extension.

Problem 1. a) Show that there is a nonzero Schwartz function f on $\mathbb{R}$ such that:

i) $f(x) \geq 0$ for all $x \in \mathbb{R}$.

ii) $\widehat{f}(t) \geq 0$ for all $t \in \mathbb{R}$.

iii) $\widehat{f}$ has compact support.

b) Show that there is no compactly supported Schwartz function f on $\mathbb{R}$ whose Fourier transform $\widehat{f}$ is also compactly supported.
Hint: The Fourier transform $\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi ixt}\mathrm{d}x$ would be holomorphic for $t \in \mathbb{C}$.

Problem 2. a) Show that there is exactly one real number $\rho > 1$ such that $\zeta(\rho) = 2$.

b) Show that there is no complex number $s \neq \rho$ with $\Re(s) \geq \rho$ and $\zeta(s) = 2$.

c) Let Δ_n be the number of ways of writing n as an ordered product of integers greater than 1:

$$\Delta_n = \#\{(k, f_1, \ldots, f_k) : \ k \geq 0, \ f_1, \ldots, f_k \geq 2, \ n = f_1 \cdots f_k\}.$$

Show that

$$\sum_{n \leq X} \Delta_n \sim -\frac{1}{\rho\zeta'(\rho)} \cdot X^\rho.$$

Problem 3. Let $a_n = 2^{-\nu(n)}$, where $\nu(n)$ is the number of primes dividing n. Show that

$$\sum_{n \le X} a_n \sim C \cdot X / \sqrt{\log X}$$

for some constant $C > 0$.

Problem 4. a) For $m \ge 0$ and any $\theta \in \mathbb{R}$, show that

$$(2m+1) + 2 \sum_{j=0}^{2m-1} (j+1) \cos((2m-j)\theta) \ge 0.$$

b) Let $f(s) = D(a,s)$ be a Dirichlet series with $a_1 = 1$. Consider the Dirichlet series $D(b,s) := -\frac{f'(s)}{f(s)}$. Assume that $b_1, b_2, \cdots \ge 0$. Furthermore, assume that both $D(a,s)$ and $D(b,s)$ have a meromorphic continuation to $\{s \in \mathbb{C} : \Re(s) \ge 1\}$. Assume that the continuation of $D(a,s)$ to this region is holomorphic except for a pole of order $e \ge 1$ at $s = 1$. Assume that the continuation of $D(b,s)$ is holomorphic in $\{s \in \mathbb{C} : \Re(s) > 1\}$. Show that $D(a,s)$ has no zero s with $\Re(s) > 1$ and that any zero s with $\Re(s) = 1$ has multiplicity at most $e/2$.