Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #1

due Wednesday, February 2 at noon

Problem 1. Here are two ways to estimate the number $N(X)$ of pairs $(x, y) \in \mathbb{N}^2$ such that $x^2 y \leq X$:

$$a)\quad N(X) = \sum_{1 \leq x \leq X^{1/2}} \sum_{1 \leq y \leq \frac{X}{x^2}} 1 \approx \sum_{1 \leq x \leq X^{1/2}} \frac{X}{x^2} \approx X \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$b)\quad N(X) = \sum_{1 \leq y \leq X} \sum_{1 \leq x \leq \left(\frac{X}{y}\right)^{1/2}} \approx \sum_{1 \leq y \leq X} \left(\frac{X}{y}\right)^{1/2} \approx X^{1/2} \cdot \sum_{1 \leq y \leq X} y^{-1/2} \approx 2 \cdot X.$$

Which of the two estimates is better for large X? Show that the better estimate differs from $N(X)$ by $C \cdot X^{1/2} + \mathcal{O}(X^{1/3})$ for some constant C.

Problem 2. Show that

$$\frac{x}{\log x} \sim \int_2^x \frac{1}{\log t} \mathrm{d}t \qquad \text{for } x \to \infty.$$

Problem 3. Show that the following two asymptotics for $x \to \infty$ are equivalent:

$$\sum_{\substack{p \leq x \\ \text{prime}}} 1 \sim \int_2^x \frac{1}{\log t} \mathrm{d}t \tag{1}$$

$$\sum_{\substack{p \leq x \\ \text{prime}}} \log p \sim x \tag{2}$$

Problem 4. Show that for any integer $m \geq 2$ and any real number $x \geq 1$,

$$\sum_{n \geq x} \exp(2\pi i n/m) \cdot 1/n = \mathcal{O}(m/x).$$

(This infinite series $\sum_{n \geq x}(\cdots)$ is not absolutely convergent, so it really means $\lim_{N \to \infty} \sum_{N \geq n \geq x}(\cdots)$.)

Problem 5 (ungraded)**.** Show that

$$\log(n!) = n \log n - n + \tfrac{1}{2} \log n + C + \mathcal{O}_{n\to\infty}(1/n)$$

for some constant $C \in \mathbb{R}$. (This turns out to be $C = \frac{1}{2}\log(2\pi)$.)

Problem 6 (bonus, ungraded)**.** Which of the following statements do you believe? Can you give a (heuristic) reason? Can you heuristically estimate the number of primes $p \leq X$ of the given form?

a) There are infinitely many primes of the form $n^2 + 1$ with $n \in \mathbb{Z}_{\geq 0}$.

b) There are infinitely many primes of the form $2^n + 1$ with $n \in \mathbb{Z}_{\geq 0}$.

c) There are infinitely many primes of the form $2^{2^n} + 1$ with $n \in \mathbb{Z}_{\geq 0}$.

Problem 7 (bonus, ungraded)**.** For which polynomials $f(X) \in \mathbb{Z}[X]$ do you expect that there are infinitely many primes of the form $f(n)$ with $n \in \mathbb{Z}$?