Math 229: Introduction to Analytic Number Theory

Spring 2022

Final exam

due Saturday, May 14 at 11:59pm (ET)

Rules: You may consult the the lecture notes (either those on the course website or those made by students), problem sets, and solutions to problem sets (including the remarks from the grader). You're not allowed to consult any other references. You may discuss the exam problems only with me (Fabian).

Honor code affirmation: Along with your solutions, please submit the following affirmation:

"I affirm my awareness of the standards of the Harvard College Honor Code."

Turning in the exam: Please submit the exam on Canvas.

You may write your solutions on the computer (e.g. using LaTeX), but clear handwriting is also accepted. If you can't fully solve a problem, still try to write down your ideas!

This exam has six problems. You can get up to 53 points in total. However, I will consider anything greater than or equal to 48 points a full score.

You may cite (without proof) any result from class or from a problem set.

Problem 1 (7 points)**.** Let $a_1, \ldots, a_n$ be positive integers. For any $d \geq 1$, we denote by r_d the number of indices i such that $d \mid a_i$. Let s be the number of indices i such that $a_i = 1$.

a) Given that $n = r_1 = 1000$ and $r_2 = 500$ and $r_3 = 300$, what is the smallest and what is the largest possible value of s?

b) What are the smallest and largest possible value of s if we additionally know that $r_6 = 100$?

Problem 2 (8 points)**.** Let $a_1, a_2, \cdots \geq 0$ be real numbers and assume that the Dirichlet series $D(a, s)$ can be meromorphically continued to (a neighborhood of) the set $\{s \in \mathbb{C} : \Re(s) \geq 0\}$, holomorphic in this region except for a simple pole at $s = 0$ with residue 1. Show that for $x \to \infty$,

$$\sum_{n \leq x} a_n \sim \log x.$$

Problem 3 (10 points)**.** Let $\nu(n)$ be the number of primes dividing n.

a) (5 points) Show that for $x \to \infty$,

$$\#\{1 \leq n \leq x \mid \nu(n) \equiv 0 \mod 2\} \sim \tfrac{1}{2}x.$$

b) (5 points) Show that for $x \to \infty$,

$$\#\{1 \leq n \leq x \mid \nu(n) \equiv 0 \mod 2 \text{ and } n \equiv 1 \mod 3\} \sim \tfrac{1}{6}x.$$

Problem 4 (10 points)**.**

a) (4 points) Let $X \geq 1$ be an integer and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Show that there are integers $1 \leq b \leq X^n$ and $a_1, \ldots, a_n$ such that $|\lambda_i - \frac{a_i}{b}| \leq \frac{1}{bX}$ for all i.

b) (6 points) Show that

$$\limsup_{t \to \infty} |\zeta(2 + it)| = \limsup_{t \to \infty} \Re(\zeta(2 + it)) = \zeta(2).$$

Problem 5 (10 points)**.** As in class, let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ for $s \in \mathbb{C}$ and $\theta(u) = \sum_{n \in \mathbb{Z}} e^{-\pi u n^2}$ for $u > 0$.

a) (2 points) Let $g(t) = \xi(\frac{1}{2} + 2\pi i t)$ for $t \in \mathbb{R}$. Show that $g(t)$ is a real number and that $g(t) = g(-t)$ for all $t \in \mathbb{R}$.

b) (2 points) For $x \in \mathbb{R}$, let

$$f(x) = e^{x/2}(\theta(e^{2x}) - 1) - e^{-x/2}.$$

Show that this function satisfies $f(-x) = f(x)$.

c) (6 points) Show that $g = \widehat{f}$, or in other words that

$$g(t) = \int_{\mathbb{R}} f(x) e^{2\pi i x t} \mathrm{d}x.$$

Remark: These facts can be used to show that $\zeta(s)$ has infinitely many zeros with $\Re(s) = \frac{1}{2}$!

Problem 6 (8 points)**.** Which of the following statements are true? Which are false? (You don't need to give a proof or counterexample.) Any correct answer for a statement gives two points. Any incorrect answer gives zero points. If you don't answer, you get one point.

a) (2 points) The abscissa of absolute convergence of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ equals the abscissa of absolute convergence of its formal derivative $\sum_{n=1}^{\infty} -a_n \log(n) n^{-s}$.

b) (2 points) If two entire functions $f, g \neq 0$ have exactly the same zeros with the same multiplicites, then $f = \lambda g$ for some $\lambda \in \mathbb{C}^{\times}$.

c) (2 points) For sufficiently large x, there are $\gg x/\log x$ integers $1 \leq n \leq x$ such that at least two of the numbers $n, n+1, \ldots, n+10^9$ are prime.

d) (2 points) For each prime number p, let S_p be a subset of $\mathbb{Z}/p\mathbb{Z}$. If $\prod_p \frac{\#S_p}{p} > 0$, then there are infinitely many integers n such that for all p, we have $n \bmod p \in S_p$.