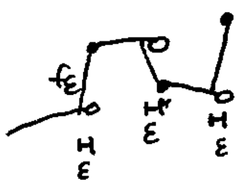
Pf 1 Apply integration by parts to the continuous extension of each $f|_{[c_i, c_{i+1})}$ to $[c_i, c_{i+1}]$ and add the results:

$$\int_a^b f'(t)g(t)\,dt + \int_a^b f(t)g'(t)\,dt = \sum_{i=0}^{n-1} \Big(\underbrace{f(c_{i+1})g(c_{i+1})}_{(f(c_{i+1})-h_{i+1})\,g(c_{i+1})} - f(c_i)g(c_i) \Big)$$

$$= f(b)g(b) - f(a)g(a) - \sum_{i=1}^{n} h_i g(c_i)$$

□

Pf 2 Apply int. by parts to f_ε, g and let $\varepsilon \to 0$.



□

Pf 3 Look up Riemann-Stieltjes integration. "□"

__Use__ • Apply liberally

• Normally, $\sum h_i g(c_i)$ is what you want to estimate.

• For best results, try making $f(x)$ small. (Usually, $\int f'(t)g(t)\,dt$ is the main term.)

• Try applying integration by parts (backwards) to $\int f g'$ after plugging in an upper bound for f.

__Example:__

__Thm 1.2.2__ Assume Thm 0.1 (PNT). Then:

$$\sum_{\substack{p\le x\\ \text{prime}}} \frac{1}{p} \sim \log\log x \quad \text{for } x\ge 2.$$

__Pf__ Use $f(x) = \sum_{\substack{p\le x\\ \text{prime}}} 1 - \int_2^x \frac{1}{\log t}\,dt = o\left(\int_2^x \frac{1}{\log t}\,dt\right)$

(jumps of height 1 at primes, $f'(x) = -\frac{1}{\log(x)}$ elsewhere)

and $g(x) = \frac{1}{x}$.

$$\int_2^x \left(-\frac{1}{\log t}\right)\cdot\frac{1}{t}\,dt + \sum_{2<p\le x} 1\cdot\frac{1}{p} + \int_2^x f(t)\cdot g'(t)\,dt = \left[f(t)\cdot\frac{1}{t}\right]_{t=a}^{b}$$

$\left[f(t)\cdot\frac{1}{t}\right]_{t=a}^{b} = O(1)$

$\int_2^x \frac{1}{t\log t}\,dt = \left[\log\log t\right]_{t=2}^{x} = \log\log x + O(1)$

Let $\varepsilon>0$. For suff. large C_ε, we have

$|f(t)| \le \varepsilon\cdot\int_2^x \frac{1}{\log t}\,dt$ for all $x\ge C$.

$$\Rightarrow \left|\int_2^x f(t)\cdot g'(t)\,dt\right| \le \underbrace{\left|\int_2^{c_\varepsilon} f(t)\, g'(t)\right|}_{=:D_\varepsilon} + \underbrace{\left|\int_2^x \left(\varepsilon\cdot\int_2^t \frac{1}{\log s}\,ds\right)\cdot g'(t)\,dt\right|}_{=:E_\varepsilon(x)}$$

$$E_\varepsilon(x) + \underbrace{\int_2^x \varepsilon\cdot\frac{1}{\log t}\cdot \underbrace{g(t)}_{\frac{1}{t}}\,dt}_{\varepsilon\cdot[\log\log t]_{t=2}^{x}} = \underbrace{\underbrace{\left[\varepsilon\cdot \underbrace{\int_2^t \frac{1}{\log s}\,ds}_{\sim \frac{t}{\log t}}\cdot\frac{1}{t}\right]_{t=2}^{x}}_{\frac{\varepsilon}{\log t}}}_{\mathcal{O}(\varepsilon)}$$

Summary:

$$\sum_{\substack{p\le x\\ \text{prime}}} \frac{1}{p} = \log\log x + \mathcal{O}_\varepsilon(1) + \mathcal{O}(\varepsilon\cdot\log\log x)$$

For any $\delta>0$, we can choose $\varepsilon>0$ so that

$$\mathcal{O}(\varepsilon\cdot\log\log x) < \frac{\delta}{2}\cdot\log\log x.$$

Then, for sufficiently large x,

$$\mathcal{O}_\varepsilon(1) < \frac{\delta}{2}\cdot\log\log x.$$

Hence, $\sum_{\substack{p\le x\\ \text{prime}}} \frac{1}{p} = \log\log x + \delta\cdot\log\log x$ for suff. large x.

In other words, $\sum_{\substack{p\le x\\ \text{prime}}} \frac{1}{p} \sim \log\log x$. $\square$

1.3. Euler-Maclaurin formulas

Def The Bernoulli polynomials $b_0, b_1, \dots$ are defined by

i) $b_0(x) = 1$

ii) $b_k'(x) = k \cdot b_{k-1}(x)$ for $k \geq 1$.

"artificial normalization"

iii) $\int_0^1 b_k(x)\,dx = 0$ for $k \geq 1$.

Ex $b_0(x) = 1$

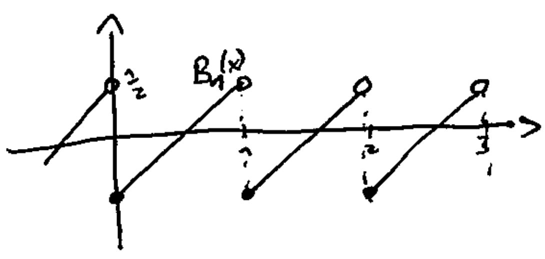
$b_1(x) = x - \frac{1}{2}$

$b_2(x) = x^2 - x + \frac{1}{6}$

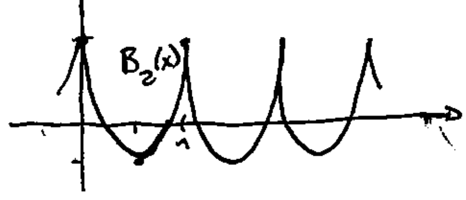
$\vdots$

Def The k-th Bernoulli function is $B_k(x) = b_k(\{x\})$.

Rmk Each B_k is periodic and in particular bounded.



Rmk $B_2, B_3, \dots$ are continuous due to iii).



Thm 1.3.1 (Euler-Maclaurin formula)

Let $k \geq 0$, and assume that $f:[a,b] \longrightarrow \mathbb{C}$ is $(k+1)$ times continuously differentiable. Then,

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\,dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!} \left[B_{r+1}(t) f^{(r)}(t) \right]_{t=a}^{b}$$

$$+ \int_a^b \frac{(-1)^k}{(k+1)!} B_{k+1}(t) f^{(k+1)}(t)\,dt$$

Ex ($k=0$) If $a,b \in \mathbb{Z}$, then $B_1(a) = B_1(b) = B_1(0) = -\frac{1}{2}$, so

$$\sum_{a\leq n\leq b} f(n) = \int_a^b f(t)\,dt + \frac{1}{2}(f(b)+f(a)) + \int_a^b B_1(t) f'(t)\,dt$$

Ex ($k=1$) If $a,b \in \mathbb{Z}$, then

$$\sum_{a\leq n\leq b} f(n) = \int_a^b f(t)\,dt + \frac{1}{2}(f(b)+f(a)) + \frac{1}{12}\left[f'(t)\right]_{t=a}^{b} - \frac{1}{2}\int_a^b B_2(t) f''(t)\,dt.$$

Rmk $\left| \int_a^b B_{k+1}(t) f^{(k+1)}(t)\,dt \right| \ll_k \int_a^b |f^{(k+1)}(t)|\,dt.$

$B_{k+1}(t) \ll_k 1$

Often, this integral is smaller / has better convergence properties for larger k.

Pf Induction over k:

k=0: Apply Abel summation to $B_1(t)$, $f(t)$:

($B_1(t)$ has jumps of height -1 at $t \in \mathbb{Z}$, $B_1'(t) = 1$ for $t \notin \mathbb{Z}$.)

$$\int_a^b 1 \cdot f(t)\,dt + \sum_{a<n\le b} (-1)\cdot f(n) + \int_a^b B_1(t) f'(t)\,dt$$

$$= \left[B_1(t) \cdot f(t) \right]_{t=a}^{b}$$

k-1 → k: Apply integration by parts to $B_{k+1}(t)$, $f^{(k)}(t)$:

(no jumps, $B_{k+1}'(t) = (k+1)\cdot B_k(t)$ for all $t \in \mathbb{Z}$)

$$\int_a^b (k+1)\cdot B_k(t) f^{(k)}(t)\,dt + \int_a^b B_{k+1}(t) f^{(k+1)}(t)\,dt$$

$$= \left[B_{k+1}(t) f^{(k)}(t) \right]_{t=a}^{b}.$$

Plug this into the induction hypothesis. $\square$

Cor 1.3.2. For $x \geq 1$,

$$\sum_{1 \leq n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

for some constant $\gamma = 0.577\ldots$ called Euler's constant or the Euler-Mascheroni constant.

Pf

$$\sum_{1 \leq n \leq x} \frac{1}{n} = \underbrace{\int_1^x \frac{1}{t}\,dt}_{\log x} + \underbrace{B_1(x)}_{O(1)} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{1} + \int_1^x B_2(t) \cdot \left(-\frac{1}{t^2}\right) dt$$

$$= \log x + \left(\frac{1}{2} + \int_1^\infty B_2(t) \cdot \left(-\frac{1}{t^2}\right) dt\right) - \underbrace{\int_x^\infty \underbrace{B_2(t)}_{O(1)} \left(-\frac{1}{t^2}\right) dt}_{O\left(\frac{1}{x}\right)} + O\left(\frac{1}{x}\right)$$

$\square$

Combining this with the improved version of (I) gives:

Thm 1.3.3 $\sum_{n \leq x} d(n) = x(\log x + 2\gamma - 1) + O(x^{1/2})$.

Pf

$$\text{LHS} = 2 \cdot \sum_{a \leq x^{1/2}} \frac{x}{a} - x + O(x^{1/2})$$

$$= 2x\left(\log x^{1/2} + \gamma + O\left(\frac{1}{x^{1/2}}\right)\right) - x + O(x^{1/2})$$

$$= \text{RHS}.$$

$\square$

Rmk It is conjectured that the error is actually only $O_\varepsilon(x^{1/4+\varepsilon})$ for any $\varepsilon > 0$.

Known (Huxley): $O_\varepsilon(x^{\frac{131}{416}+\varepsilon})$.