Math 137: Algebraic Geometry

Spring 2022

Problem set #8

due Friday, April 15 at noon

Throughout, K is assumed to be an algebraically closed field.

Warning: Corollary 13.4.3 as stated in class is false. (But, as we'll see on Monday, it holds for $W = K^n$. Sorry...)

Problem 1. We call $P \in K^n$ a *point of symmetry* for a subset $S \subseteq K^n$ if the reflection $2P - Q$ across P of any point $Q \in S$ lies in S. Assuming that K has characteristic zero, show that any nonempty algebraic subset $S \subseteq K^n$ that doesn't contain a straight line has at most one point of symmetry.

Problem 2. Let $\varphi : V \to W$ be a dominant morphism between irreducible algebraic sets. Assume that there is a nonempty Zariski open subset U of W such that $|\varphi(w)| < \infty$ for all $w \in U$. Show that $\dim(V) = \dim(W)$.

Problem 3. For $r \leq n$, consider the set $V_r \subseteq M_n(K)$ of $n \times n$-matrices of rank at most r. You've shown on problem set 3 that V_r is an algebraic subset of $M_n(K) = K^{n \times n}$. Show that its dimension is $2nr - r^2$.

Problem 4. Let $V \subseteq K^n$ be an irreducible algebraic set and let $P \in K^n$ be a point not contained in V. Show that the Zariski closure of the join of V and $\{P\}$ has dimension $\dim(V) + 1$.

Problem 5 (bonus)**.** Let $V_1, \ldots, V_m$ be any irreducible algebraic subsets of K^n of codimension at least 2. Show that there is an irreducible algebraic subset $W \subsetneq K^n$ containing $V_1 \cup \cdots \cup V_m$.
Hint: What is the dimension of the space of polynomials of degree at most d vanishing on $V_1 \cup \cdots \cup V_m$? What is the dimension of the space of polynomials that are not irreducible?

Problem 6. Let $n \geq 2$ and $d \geq 1$. Consider the vector space $F_d \cong K^{\binom{n+d}{n}}$ of polynomials f in $K[X_1, \ldots, X_n]$ of degree at most d. Show that there is a function $0 \neq r \in \Gamma(F_d)$ (a polynomial in the $\binom{n+d}{n}$ coefficients of f) such that $r(f) = 0$ for all reducible polynomials $f \in F_d$.